Universidade de Aveiro Departamento de Matemática
2008

Ricardo Miguel Moreira de Almeida

Fundamentos para a Análise Não-standard Não Linear

Foundations of Nonstandard Non-linear Analysis

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Foundations of Nonstandard Non-linear Analysis

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Vítor Manuel Carvalho das Neves, Professor Associado do Departamento de Matemática da Universidade de Aveiro.

Thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of *Doctor of Philosophy* in Mathematics, under the supervision of Vítor Manuel Carvalho das Neves, Associated Professor at the Department of Mathematics of the University of Aveiro.

o júri

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palavras-chave

Análise Não-Standard, Conexão, Teorema do Valor Médio, Diferenciabilidade, Curvas, Superfícies Regulares, Variedades Diferenciais

resumo

Esta tese insere-se na área da análise não-standard não linear. São dois os objectivos principais deste trabalho. Um deles envolve diferenciabilidade de funções e o outro geometria diferencial.
O nosso trabalho é dividido em três partes.
Na primeira apresentamos uma caracterização não-standard de conjuntos compactos conexos em espaços métricos.
Na segunda parte exibimos alguns resultados envolvendo o teorema do valor médio para espaços normados. De seguida é apresentado um novo tipo de diferenciabilidade, a mu-diferenciabilidade. Fazemos um estudo exaustivo das propriedades básicas deste tipo de derivada, nomeadamente a regra em cadeia, o teorema do valor médio, o teorema de Taylor e o teorema da função inversa.
A última parte é dedicada à geometria diferencial. Primeiro em espaços de dimensão finita, com uma caracterização não-standard de cúspides e de superfícies regulares. Também apresentamos uma construção de uma curva interna que seria uma solução ideal para um problema de máxima resistência.
Por último apresentamos um estudo de variedades diferenciáveis em espaços de Banach. O análogo de espaço tangente, derivada de uma função e derivadas direccionais são apresentados.

keywords

Nonstandard Analysis, Connected Sets, Mean Value Theorem, Differentiability, Curves, Regular Surfaces, Differentiable Manifolds

abstract

This thesis presents a study of non-linear nonstandard analysis. There are two main goals of this study. One of them concerns differentiability of functions and the other one is differential geometry.

Our study is divided in three parts.

The first part contains a nonstandard characterization of connected compact sets in metric spaces.

In the second part we give some results related to the mean value theorem in normed spaces. Then we describe a new type of differentiability, the mu-differentiability. We carry out an extensive study of the basic properties of this differentiation, namely the chain rulle, the mean value theorem, Taylor's theorem and the inverse mapping theorem.

The final part is devoted to differential geometry. First in finite dimensional spaces, with nonstandard characterizations of cusps and regular surfaces. We also construct an internal curve which would be an ideal solution for a problem of maximum resistance. Finally we deal with differentiable manifolds modeled on Banach spaces. The analogous of tangent bundle, differential of a function and directional derivatives are given.

Nenhum homem é uma ilha isolada;
cada homem é uma partícula do continente, uma parte da terra;
se um torrão é arrastado para o mar, a Europa fica diminuída,
como se fosse um promontório como se fosse a casa dos teus amigos ou a tua própria;
a morte de qualquer homem diminuiu-me, porque sou parte do género humano.
E por isso não perguntes por quem os sinos dobram;
Eles dobram por ti

Jonh Donne

Contents

Basic notation:

$:=$	equality, by definition;
$\emptyset$	empty set;
$\mathbb{N}$	set of natural numbers;
$\mathbb{Z}$	set of integer numbers;
$\mathbb{Q}$	set of rational numbers;
$\mathbb{R}$	set of real numbers;
$\mathbb{R}^+$	set of positive real numbers;
$\mathbb{R}_0^+$	set of nonnegative real numbers;
$\mathbb{R}^n$	n-dimensional Euclidean space;
$B_r(a)$	open ball centered at a and radius $r \in \mathbb{R}^+$;
$\overline{B}_r(a)$	closed ball centered at a and radius $r \in \mathbb{R}^+$;
$card(E)$	cardinality of the set E;
$\mathcal{P}(E)$	power set of E;
B^A	set of all functions from the set A into the set B;
$f_{\vert A}$	restriction of the function $f : X \to Y$ to $A \subseteq X$;
$\cdot$	takes the place of the variable with respect to which the function is evaluated;
$\vert \cdot \vert$	norm;
$a \cdot b$	inner product between a and b;
$\blacksquare$	end of proof.

Other symbols and pages where they are introduced:

Introduction

Overview

Let's start at the very beginning,
A very good place to start,
When you write you begin with A,B,C
When you differentiate you begin with "given ϵ there is a δ".
Adrian P. Simpson, *The infidel is innocent* [Sim90]

Nonstandard Analysis (NSA) (or the Theory of Infinitesimals as some prefer to call it) was invented by Abraham Robinson in the 1960's, and among other things, provided an answer to an old question: Do the infinitesimals, as understood by Leibniz and Newton, exist as mathematical objects? As he wrote in the preface to his book *Non-standard Analysis* [Rob74]:

In the fall of 1960 *it occurred me that the concepts and methods of contemporary Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers.*

He showed that we can embed the ordered field of real numbers $(\mathbb{R}, +, \cdot, \leq)$ as an ordered subfield of a structure $({}^*\mathbb{R}, {}^*{+}, {}^*{\cdot}, {}^*{\leq})$ (the set of hyper-real numbers) which, besides being a totally ordered field, contains other numbers such as infinitesimal numbers and infinitely large numbers. There is no controversy concerning the logical soundness of hyper-real numbers. The use of infinitesimals in the early development of calculus beginning with Leibniz, continuing with Euler, and persisting to the time of Gauss was problematic. The founders knew that their use was logically incomplete and could lead to incorrect results. Hyper-real numbers are a correct treatment of infinitesimals that took nearly 300 years to discover. Leibniz apparently considered his idealized numbers as the ordinary reals with some infinitesimals clustered around zero. As we shall see, we can look to finite hyper-reals as ordinary reals with new numbers clustered infinitesimally closely around each ordinary real. Our intuition about the properties of such numbers can be formulated and proved, and this allows classical analysis to be developed rigorously in a natural way with the aid of these numbers and concepts. Moreover, all the valid sentences in the real structure continue to be valid in the hyper-real structure (the **Transfer Principle**). It is easily seen that the proper extension field $^*\mathbb{R}$ cannot satisfy all the properties of $\mathbb{R}$. For example, the set of finite numbers in $^*\mathbb{R}$ is bounded from above by any positive infinite number, but cannot have a least upper bound. The challenge was to establish a clear and consistent foundation for dealing with infinitesimals, that capture the known heuristic arguments as much as possible.

This has enabled us to return to the more intuitive analytical approach of the originators of the calculus. The presence of infinitesimals allows us to give elegant and useful characterizations of many important mathematical concepts. For example, one can prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ if and only if for all infinitesimal ϵ, $f(x + \epsilon) - f(x)$ is infinitesimal.

Jerome Keisler wrote once that NSA *"will probably rank as one of the major mathematical advances of the twentieth century."* Since then, NSA has been applied to several problems in mathematics such as in Banach spaces, differential equations, probability theory, algebraic number theory, stochastic analysis, functional analysis, and in other fields such as physics and economics. It has also been suggested that NSA should be used in secondary and university education to help the student reach a deeper understanding of analysis (see [Kei86], [Tod01], [OK06] and [O'D07]). In short, NSA provides us with an enlarged view of the mathematical landscape with some advantages; the corresponding richer nonstandard theory proves to be more intuitive and thus easier to deal with as well.

Our work has two main areas, differential calculus and differential geometry, and both are strongly associated with the idea of an infinitesimal number.

We begin by studying connectedness in metric spaces. Although there are some appreciable results in topology (for example, the nonstandard characterization of compactness is wonderful) no one has done the same with connectedness. In [Let95], Steven Leth presents a sufficient condition for a set $A \subseteq \mathbb{R}^n$ to be connected. But, as the author remarked, it is not a necessary condition. His hypothesis involves internal polygonal paths joining distinct points. We will work with hyper-finite sets instead with a polygonal, thus eliminating the implicitly present local path-connectedness that there is on his research. We mention also the work of Sérgio Rodrigues [Rod01], where he characterizes connectedness in nonstandard terms, using the monad of the set .

The derivative of a real function measures the ratio between $f(a+\epsilon)-f(a)$ and ϵ, when $\epsilon \to 0$. But this means that ϵ is actually a non-zero infinitesimal number. The problem is that on classical analysis, such things as non-zero infinitesimals do not exist. So solve it, we begin by choosing an $\epsilon \in \mathbb{R}$, evaluate the quotient

$$\frac{f(a+\epsilon)-f(a)}{\epsilon}$$

and afterwards we calculate the limit when ϵ tends to zero. Using NSA language, we are allowed to use these new numbers, and so obtain a simple characterization of the derivative. In the literature we do not found much work related to differentiability of a function using nonstandard techniques, on in their own right. Basically we have the nonstandard analog to Fréchet derivative and nonstandard proofs of the basic theorems like the Chain Rule, Taylor's Theorem, Inverse Mapping Theorem and so on (see [SL76] and [Str78]).

We dedicate a chapter of the thesis to study the Mean Value Theorem in normed spaces. A simple proof can be done, and we will see that the point c that we must ensure the existence is actually a point where the derivative is infinitely close to a mean of derivatives at some points $x_1 < \ldots < x_n$,

$$f'(c) \approx \frac{f'(x_1)+\ldots+f'(x_n)}{n}$$

Some results related to the theorem are presented.

In [Sch97], Reeken presents a new type of derivative, what he called m-derivative. For classical functions, it is simply the Fréchet derivative. But for internal functions, it seems to have some advantages, namely in physics. In this book, Reeken just presents the definition and then establish a necessary and sufficient condition to test m-differentiability (Theorem 4.2). In our work we go further. First we extend his definition to uniform m-differentiability and then we present the basic theorems for this type of derivative.

Differential geometry also appears to be a natural field of application of NSA. Knowledge of curvature, tangent vector or tangent plane to a regular surface, require only properties of the curve, surface, etc, on an infinitely small neighbourhood of each point. In [HJ01], Hertrich-Jeromin presents a nonstandard characterization of submanifolds in Euclidean spaces, that we will latter use to characterize regular surfaces. More can be found with a different flavour in [HJ00] and in [Str77], where the authors treat basic infinitesimal geometry.

In chapter five we present and relate several definitions of cusp, all so to speak geometrically evident. Somewhat surprisingly none presupposes any kind of differentiability of the curve. Under some conditions, they are all actually equivalent.

A section of this chapter is dedicated to a direct presentation of a curve of infinitely large resistance as opposed to the approximation procedure described in [Pla06]: therein Alexandre Plakhov defines a family of sets $\{\Omega_\epsilon \subseteq \mathbb{R}^2 \mid \epsilon > 0\}$ which produce better and better boundaries as $\epsilon \to 0$.

In the last chapter we stay inside the category of classical manifolds. Using nonstandard analysis techniques, we present some new definitions for the tangent bundle and the differential of a function.

For a different approach concerning manifolds, the reader may consult [Sch97]. Here the author uses the nonstandard methods as a pathway to a generalization of differential geometry, by developing a theory of a nonstandard analog of manifolds. Two concepts of manifolds are presented and related. Analogs to tangent bundle and Riemannian structure are also given.

Organization of the Dissertation

The aim of this thesis is to present some applications of NSA in several fields of mathematics.

The **first chapter** of this work introduces the reader to NSA. It is not our intention to give many details on the subject, so basically we will just fix some necessary terminology and some important facts, such as the **Transfer Principle** or the **Spillover Principle**. The rest of the chapter is devoted to the study of topological spaces. We have also included a brief discussion on metric and normed spaces; in particular, we define S-differentiability and relate it with the Fréchet differentiation.

In the rest of the work we present our original results. The major contributions of this dissertation are summarized next.

In **Chapter 2** we present some sufficient conditions for continuity and with them we prove two known theorems on continuity of functions. The next section is devoted to connectedness and compactness on standard sets; we introduce a new concept, the *discrete infinitesimal path*, which will be used to characterize connected compact sets in metric spaces.

In **Chapter 3**, we first give a nonstandard proof of the Mean Value Theorem (**MVT**) which uses only the Intermediate Value Theorem for real functions with real variable. We also obtain a generalization of the **MVT** for internal SU-differentiable functions. Next we generalize some theorems involving the **MVT** presented in [Jac82], [Bl97] and [TB97], namely we present an estimation for the differential mean point and also a converse of the **MVT**.

Following Reeken's ideas, in **Chapter 4** we present a new concept, the *macroscopic-uniform differentiability* (abbreviated mu-differentiability). We give a necessary and sufficient condition for an internal function to be mu-differentiable, relating with a standard C^1 function. The Chain Rule, Taylor's Theorem, Mean Value Theorem and an Inverse Mapping Theorem are also proved for mu-differentiable functions.

Chapter 5 is the geometric core of the work. Several nonstandard definitions of cusps are presented and we establish relations between them as well with the classical definitions of cusps.

In addition we present a new method to determine the envelope to a family of C^1 curves. Later we will apply them to the well known problem of the coffecup caustic. Afterward we define an internal curve which will be an "ideal" solution for a problem on maximum resistence. At the end we present a nonstandard characterization of regular surfaces.

Chapter 6 is entirely devoted to differentiable manifolds. In [SL76] is introduced the tangent bundle to a differentiable manifold M by means of a kind of vector field on M, $X : M \to M$. We extend their work, presenting a new definition of tangent space, and some original results are formulated using those internal functions X.

At the beginning there is a list of the notation, together with the page where the notation is introduced.

Future Work

Some open problems remain in our work:

1. Present a nonstandard characterization of connectedness in arbitrary topological spaces.

2. To give a nonstandard proof of the converse of the Mean Value Theorem.

3. Is the Chain Rule as such valid for mu-differentiable functions?

4. Is the definition of mu-differentiable function equivalent to

   There exists $0 \approx \delta > 0$ such that, for each $x \in ns({}^*U)$, there exists a finite linear operator $Df_x \in {}^*L(E, F)$ for which holds

   $$\forall y \approx x \;\; |x-y| > \delta_a \Rightarrow f(x) - f(y) = Df_x(x-y) + |x-y|\eta$$

   for some $\eta \approx 0$?

5. To give Palais-Smale conditions for $f : M \to \mathbb{R}$ using the δ-infinitesimal transformations, where M is a standard manifold.

Chapter 1

The Nonstandard Universe

This thesis is intended to be an exposition of applied Nonstandard Analysis (NSA): we are interested in the theory as a tool for studying mathematical structures. This will reflect in this introduction: we will assume that the reader accepts the fact that, given any set S large enough to contain the elements we work on (complex numbers, topological spaces, Banach spaces, etc), it has a nonstandard extension ${}^*S$ containing new *ideal* elements, *e.g.*, infinitesimals in ${}^*\mathbb{R}$. For further details, see [HL85], [SL76], [DD95], [CE88] or [LG81].

The first full version of NSA was presented in 1966 by Robinson. This version relied on a certain familiarity with Mathematical Logic, and in particular, Model Theory. More recently, by placing NSA in the framework of Superstructures, much of the technical aspects related to Mathematical Logic have been dispensed with. In this thesis we shall follow this approach.

This chapter is designed as an introduction to NSA, giving an exposition of the foundations of the theory and some results needed for the present work. We will omit most proofs and technical details.

1.1 Nonstandard Analysis on Superstructures

Definition 1.1 *Let S be a nonempty set. As usual, we represent $\mathcal{P}(S)$, the power set of S, the set of all subsets of S. Define the* **nth cumulative power set** *$V_n(S)$ of S by the following induction on n:*

$$\begin{aligned} V_0(S) &:= S \\ V_{n+1}(S) &:= V_n(S) \cup \mathcal{P}(V_n(S)). \end{aligned}$$

The ***superstructure*** *of S is the union $\bigcup_{n=0}^{\infty} V_n(S)$ and is denoted by $V(S)$.*

Notice that we have the inclusions

$$S = V_0(S) \subset V_1(S) \subset V_2(S) \subset \dots$$

and also

$$V_j(S) \in V_k(S) \text{ whenever } j < k.$$

Let Φ be a formula in a first order language $\mathcal{L}_S$ about mathematical entities in S. The **$*$-transformation** ${}^*\Phi$ of Φ is the formula in another first order language $\mathcal{L}_{^*S}$ about mathematical entities in ${}^*S$ (see page 7) obtained from Φ by replacing each constant symbol c in Φ with the symbol ${}^*c$ in $\mathcal{L}_{^*S}$. For example, the $*$-transformation of the sentence (obviously, abbreviated)

$$\forall x, y \in \mathbb{R}\, [x < y \Rightarrow \exists q \in \mathbb{Q}\, [x < q < y]]$$

is the sentence

$$\forall x, y \in {}^*\mathbb{R}\, [x \,{}^*\!\!< y \Rightarrow \exists q \in {}^*\mathbb{Q}\, [x \,{}^*\!\!< q \,{}^*\!\!< y]].$$

This is an instance of Transfer in Nonstandard Analysis, a two way interaction between different mathematical structures, the standard and nonstandard theories.

Theorem 1.2 ***Transfer Principle*** *[HL85] For any sentence Φ in $\mathcal{L}_S$, Φ holds in $\mathcal{L}_S$ if and only if ${}^*\Phi$ holds in $\mathcal{L}_{^*S}$.*

For instance, we know that

$$\forall x \in \mathbb{R}\, \exists n \in \mathbb{N}\ \ x < n.$$

So, by the Transfer Principle

$$\forall x \in {}^*\mathbb{R}\, \exists n \in {}^*\mathbb{N}\ \ x \,{}^*\!< n.$$

When using the Transfer Principle we must replace **all** constant symbols c by ${}^*c$. For example, the sentence

$$\forall x \in {}^*\mathbb{R}\, \exists n \in \mathbb{N}\ \ x \,{}^*\!< n.$$

is false.

The Transfer Principle is an often used very powerful tool in NSA. For example, if f is a function from a to b, then ${}^*f$ is a function from ${}^*a$ to ${}^*b$ and ${}^*[f(c)] = {}^*f({}^*c)$, for each $c \in a$.

To simplify, from now on we will write $\leq$, $=$, f, etc instead of ${}^*\!\leq$, ${}^*\!=$, ${}^*f$, etc.

Definition 1.3 *All entities in $V(S)$ are called **standard**. Also, if $b \in V({}^*S)$ and $b = {}^*a$, for some $a \in V(S)$, then b is also called **standard**; b is said to be the **nonstandard extension** of a. All other entities are called **nonstandard**.*

*An entity $b \in V({}^*S)$ is called **internal** if $b \in {}^*a$, for some $a \in V(S)$, i.e., internal entities are elements of standard entities. Otherwise, we say that b is **external**. In an analogous way, a formula in $\mathcal{L}_{{}^*S}$ is called **standard** or **internal** if the constants appearing in the formula are standard or internal entities, respectively. A sentence which is not internal is **external**.*

If $\Phi(x)$ is an internal formula in $\mathcal{L}_{{}^*S}$ for which x is the only free variable and A is an internal set, then $\{x \in A \,|\, \Phi(x) \text{ is true}\}$ is internal (*Keisler's Internal Definition Principle*, see [HL85]).

Definition 1.4 *Given a standard set A, define the **standard copy** of A by*

$${}^\sigma A := \{{}^*a \,|\, a \in A\}.$$

We will often write simply A instead of ${}^\sigma A$ when there is no danger of mislead. When A is infinite, ${}^\sigma A$ is a proper subset of ${}^*A$. As a matter of fact,

$$^{\sigma}A \text{ is internal if and only if } A \text{ is finite } i.e., \ {}^{\sigma}A = {}^{*}A.$$

Definition 1.5 *Let A be a set. We say that A if* ***finite*** *(resp.* ***hyper-finite****) with cardinality $n \in \mathbb{N}$ (resp. $n \in {}^{*}\mathbb{N}$) if there exists an internal bijection $f : \{1, \dots, n\} \to A$. The number n is called the* ***internal cardinality*** *of A.*

Theorem 1.6 *[Lin88] Every hyper-finite nonempty subset of ${}^{*}\mathbb{R}$ has a maximum and a minimum.*

It is also true that

Theorem 1.7 ***Discretization Principle*** *[Nev01] For any standard set X there exists a hyper-finite set $\mathcal{H}$ such that*

$$^{\sigma}X \subseteq \mathcal{H} \subseteq {}^{*}X.$$

Furthermore, X is infinite if and only if both inclusions are strict.

Theorem 1.8 ***Comprehension Principle*** *[SL76] Suppose that X and Y are sets in $V(S)$, $A \subseteq {}^{*}X$, $B \subseteq {}^{*}Y$, $card(A) < card(\mathcal{X})$ and B is internal. For each $f : A \to B$ there exists an internal function $g : {}^{*}X \to B$ such that $g|_A = f$.*

1.2 Topology

We will now present some theorems in Topology; instead neighbourhoods, monads (see below) will be used to define openness, continuity, etc. The fundamental idea here is that the notion of **infinitely close** can thus be made precise for any topological setting.

Let $(X, \mathcal{T})$ be a topological space.

Definition 1.9 *For a fixed point $x \in {}^{\sigma}X$, the* ***monad*** *of x is the subset of ${}^{*}X$ given by*

$$\mu(x) := \bigcap \{ {}^{*}O \mid x \in O \wedge O \in \mathcal{T} \}.$$

*A point $y \in {}^*X$ is **nearstandard** if there exists some $x \in {}^\sigma X$ with $y \in \mu(x)$; in this case we say that x is the **standard part** of y and write $st(y) = x$. We also say that x is **infinitely close** to y and write $x \approx y$. Two points $x, y \in {}^*X$ are infinitely close if they belong to the same monad. If x and y are not infinitely close we write $x \not\approx y$.*

The set of the nearstandard points of ${}^*X$ is

$$ns({}^*X) := \bigcup\{\mu(x) \mid x \in {}^\sigma X\}.$$

Note also that we only defined monads for standard points in ${}^*X$; hence the relation $\approx$ is not necessarily an equivalence relation on ${}^*X$.

Theorem 1.10 *[HL85] Let $A \subseteq X$. Then*

1. *A is open if and only if for all $a \in A$, $\mu(a) \subseteq {}^*A$ holds;*
2. *A is closed if and only if, whenever $a \in {}^*A$ and $a \approx x$ for some $x \in X$, $x \in A$;*
3. *A is compact if and only if for all $a \in {}^*A$ there is $x \in A$ with $a \approx x$;*
4. *$x \in X$ is an accumulation point of A if and only if there exists $a \in {}^*A$ with $x \neq a$ and $x \approx a$;*
5. *$(X, \mathcal{T})$ is Hausdorff if and only if monads of distinct points in X are disjoint.*

Observe that the standard part of an element, if it exists, is not necessarily unique. For example, for the trivial topology, *i.e.*, when $\mathcal{T} = \{\emptyset, X\}$, we have $\mu(x) = {}^*X$.

By Theorem 1.10, if X is a Hausdorff space then for all $x \in ns({}^*X)$, there exists exactly one element in X infinitely close to x. In that case we have a well-defined function

$$\begin{array}{rccc} st: & ns({}^*X) & \to & X \\ & x & \mapsto & st(x) \end{array}$$

called the **standard part function**.

Theorem 1.11 *[HL85] Let X and Y be two topological spaces and $f : X \to Y$ a function. Then f is continuous if and only if*

$$\forall x \in X \quad f(\mu(x)) \subseteq \mu(f(x)),$$

or equivalently,

$$\forall x \in X \, \forall y \in {}^*X \quad [x \approx y \Rightarrow f(x) \approx f(y)].$$

In particular, with the normed space $\mathbb{R}$, we establish the following.

Definition 1.12 *Let $x, y \in {}^*\mathbb{R}$. We say that*

1. *x is **infinitesimal** if $|x| < \epsilon$, for all positive real number ϵ and we write $x \approx 0$;*
2. *x is **finite** if, for some positive real number ϵ, $|x| < \epsilon$;*
3. *x is **infinite** (or **infinitely large**) if it is not finite, i.e., for all positive real number ϵ, $|x| > \epsilon$ and write $x \approx \infty$;*
4. *x, y are **infinitely close** if $x - y$ is infinitesimal and we write $x \approx y$.*

1.3 Metric and Normed Spaces

Let (X, d) be a metric space. We can formulate, even generalize, the previous concepts using the metric d. For example, two points $x, y \in {}^*X$ are infinitely close if $d(x, y) \approx 0$; the set $\mu(x)$ for $x \in {}^*X$ is $\{y \in {}^*X \mid x \approx y\}$. For metric spaces, $\approx$ is an equivalence relation on ${}^*X$.

The intuitive notion of continuity of a function is that a small change in the independent variable produces a small change on the image. We can express *standard continuity* and *standard uniform continuity* by

Definition 1.13 *Let X and Y be two metric spaces and $f : {}^*X \to {}^*Y$ be an internal function. We say that*

1. *f is* ***S-continuous*** *if for all $x \in {}^\sigma X$ and $y \in {}^*X$, if $x \approx y$ then $f(x) \approx f(y)$;*

2. *f is* ***SU-continuous*** *if for all $x, y \in {}^*X$, if $x \approx y$ then $f(x) \approx f(y)$.*

Theorem 1.14 *[HL85] A standard function f is continuous (resp. uniformly continuous) if and only if it is S-continuous (resp. SU-continuous).*

For instance, $f(x) = x^2, x \in \mathbb{R}$ is not uniformly continuous since, if ω is an infinite hyper-real number then

$$f\left(\omega + \frac{1}{\omega}\right) = \omega^2 + \frac{1}{\omega^2} + 2 \not\approx \omega^2 = f(\omega).$$

NSA does simplify the proofs of a significant number of classical results. For example, we will prove the next theorem.

Theorem 1.15 *[HL85] If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.*

Proof. Fix $x, y \in {}^*X$ with $x \approx y$. Compactness of X guarantees the existence of $z \in X$ with $x \approx z \approx y$. Using the continuity of f, $f(x) \approx f(z) \approx f(y)$, as desired. ■

Given $x \in {}^*X$, we say that x is **pre-nearstandard** if

$$\forall \epsilon \in {}^\sigma\mathbb{R}^+ \, \exists y \in {}^\sigma X \;\; d(x, y) < \epsilon.$$

The set of pre-nearstandard points of ${}^*X$ will be denoted by $pns({}^*X)$. It is clear that $ns({}^*X) \subseteq pns({}^*X)$.

Theorem 1.16 *[HL85] A metric space (X, d) is complete if and only if $pns({}^*X) = ns({}^*X)$.*

For example, one way verifying that $\mathbb{Q}$ with the Euclidean metric is not complete is the following. Say

$$\sqrt{2} = \sum_{n=0}^{\infty} \frac{a_n}{10^n}$$

with $0 \leq a_n \leq 9$. By transfer,

$$\sqrt{2} = \sum_{n \in {}^*\mathbb{N}_0} \frac{a_n}{10^n}.$$

Let $\omega \in {}^*\mathbb{N}$ be an infinite number and define $q := \sum_{n=0}^{\omega} \frac{a_n}{10^n}$. Then $q \in pns({}^*\mathbb{Q})$. In fact, let $\epsilon = 10^{-N}$, for some $N \in \mathbb{N}$, then

$$\left| \sum_{n=0}^{\omega} \frac{a_n}{10^n} - \sum_{n=0}^{N+1} \frac{a_n}{10^n} \right| < \epsilon$$

and $\sum_{n=0}^{N+1} \frac{a_n}{10^n}$ is a standard rational number. But $q \approx \sqrt{2}$ and so $q \notin ns({}^*\mathbb{Q})$.

Finally we reach normed spaces. We will now present the definition of infinitesimal and infinitely large vectors.

Definition 1.17 *Let x and y be two vectors of a normed space $({}^*E, |\cdot|)$. We say that*

1. *x is **infinitesimal** if $|x|$ is infinitesimal; the set of infinitesimals will be denoted by $inf({}^*E)$;*

2. *x is **finite** if $|x|$ is finite;*

3. *x is **infinite** if $|x|$ is infinite and write $x \approx \infty$;*

4. *x and y are **infinitely close** if $x - y$ is infinitesimal and we write $x \approx y$.*

The set of finite vectors of ${}^*E$ is represented by $fin({}^*E)$ (the set of nearstandards is still denoted by $ns({}^*E)$ as before).

The set of infinitesimal vectors is the monad of zero. The set ${}^*\mathbb{N}_\infty$ represents the positive integers infinitely large, ${}^*\mathbb{N}_\infty = {}^*\mathbb{N} \setminus {}^\sigma\mathbb{N}$. In an analogous way we could define ${}^*\mathbb{Z}^+_\infty$,${}^*\mathbb{Z}^-_\infty$,${}^*\mathbb{Z}_\infty$, ${}^*\mathbb{R}_\infty$, etc.

Theorem 1.18 *[Dav] If E is a normed space then $ns({}^*E) \subseteq fin({}^*E)$. Moreover, E is finite dimensional if and only if $ns({}^*E) = fin({}^*E)$.*

In fact, if E is an infinite dimensional space and x a finite vector, we can not conclude that x is nearstandard. For example, take $x = (x_n)$, $(n \in {}^*\mathbb{N})$ where

$$x_n = \begin{cases} 0 & \text{if} \quad n \neq \omega \\ 1 & \text{if} \quad n = \omega \end{cases}$$

and $\omega \in {}^*\mathbb{N}_\infty$. Then $x \in {}^*l^1(\mathbb{R})$, x if finite ($|x| = 1$) but is not infinitely close to any standard sequence.

Theorem 1.19 *[HL85] Given a sequence $(x_n)_n$ in E, we have that*

1. *$(x_n)_n$ is bounded if and only if $x_n \in fin({}^*E)$, for all $n \in {}^*\mathbb{N}_\infty$;*
2. *$(x_n)_n$ converges to $x \in E$ if and only if $x_n \approx x$, for all $n \in {}^*\mathbb{N}_\infty$;*
3. *$(x_n)_n$ has a convergent subsequence if and only if $x_n \in ns({}^*E)$, for some $n \in {}^*\mathbb{N}_\infty$;*
4. *$(x_n)_n$ is a Cauchy sequence if and only if $x_n \approx x_m$, for all $n, m \in {}^*\mathbb{N}_\infty$.*

Theorem 1.20 ***Spillover Principle*** *[HL85] Let A be an internal subset of ${}^*\mathbb{R}$.*

1. ***Overflow Principle*** *If A contains all standard positive hyper-real numbers, then A contains a positive infinite number.*
2. ***Underflow Principle*** *If A contains all infinite positive hyper-real numbers, then A contains a positive standard number.*
3. ***Local Overflow Principle*** *If A contains all positive infinitesimal numbers, then A contains a positive standard number.*

Let A be a subset of E. In the following we will denote

$$ns({}^*A) := \{x \in {}^*A \,|\, x \in ns({}^*E) \wedge st(x) \in {}^\sigma A\}$$

Given an internal linear operator $L \in {}^*L(E,F)$, we say that L is finite if $L(fin({}^*E)) \subseteq fin({}^*F)$.

Definition 1.21 *Let E and F be two normed spaces, U an open subset of E and $f : {}^*U \to {}^*F$ an internal function with $f(ns({}^*U)) \subseteq ns({}^*F)$. Then*

1. *f is **S-differentiable** if for all $x \in U$, there exists an internal finite linear operator from E into F, $L_x \in L(E,F)$, such that whenever $y \approx x$, there exists an infinitesimal number η satisfying*
$$f(x) - f(y) = L_x(x-y) + |x-y|\eta;$$
2. *f is **SU-differentiable** if for all $x \in ns({}^*U)$, there exists an internal finite linear operator $L_x \in {}^*L(E,F)$ such that, whenever $y \approx x$, there exists an infinitesimal number η satisfying*
$$f(x) - f(y) = L_x(x-y) + |x-y|\eta.$$

When such L_x exists we shall denote it by Df_x. We should remark that in contrast to classical differentiability, the linear operator Df_x involved in the definition of the nonstandard analogs are necessarily non-unique since an infinitesimal variation produces an equally well suited one.

Observe also that if $x \in ns({}^*U)$ and $y \approx x$, then $y \in {}^*U$ since U is an open set.

If f is a standard function then f is differentiable or continuously differentiable if and only if f is S-differentiable or SU-differentiable, respectively (see [SL76] or [Str78]).

Theorem 1.22 *[SL76] An internal function $f : {}^*U \to {}^*F$ is SU-differentiable if and only if for all $a \in {}^\sigma U$, there exists an internal finite linear operator $L_a \in L(E,F)$ such that, whenever $y \approx x \approx a$, there exists an infinitesimal number η satisfying*
$$f(x) - f(y) = L_a(x-y) + |x-y|\eta.$$

One of the classical results often used it is *Taylor's Theorem.* There is a nonstandard version of that theorem that we present now. This theorem is very powerful since it provides us a necessary and sufficient condition for a function to be of class C^k.

Theorem 1.23 ***Taylor's Theorem*** *[SL76] Let $f : U \to F$ be a function. Then f is of class C^k if and only if there exist unique maps $L^h_{(.)} : U \to SLin^h(E, F)$, $h \in \{1, \ldots, k\}$ (where $SLin^h(E, F)$ denotes the symmetric h-linear operators from $E \times \ldots \times E = E^h$ into F) such that, whenever $a \in ns({}^*U)$ and $x \approx a$, there is an infinitesimal $\eta \in {}^*F$ satisfying*

$$f(x) = \sum_{h=0}^{k} \frac{1}{h!} L^h_a (x-a)^{(h)} + |x-a|^k \eta.$$

The unique maps $L^h = D^h f$.

Chapter 2

NSA and Topology

In this chapter we will study some applications of NSA to topology. Namely, we will present some results on continuity of functions and give a nonstandard characterization of connected compact sets.

2.1 Continuity

Let E be a linear space. Recall that a function $f : E \to \mathbb{R}$ is called **convex** if

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \qquad \text{(Jensen's inequality)}$$

for all $x_1, x_2 \in E$ and $\lambda \in]0, 1[$.

Theorem 2.1 *[Alm07b] Let $(E, |\cdot|)$ be a normed space and $f : E \to \mathbb{R}$ a convex function. If $f({}^*S^1) \subseteq fin({}^*\mathbb{R})$, where S^1 denotes the unit sphere in E, then f is continuous.*

Proof. Fix $x_0 \in E$. Without any loss of generality, we may assume that $x_0 = 0$ and $f(x_0) = 0$ (simply replace f by the convex function $g(x) := f(x + x_0) - f(x_0)$). Then given $0 \approx \epsilon \in {}^*E$ with $\epsilon \neq 0$, we have

1. $f(\epsilon) \lesssim 0$, *i.e.*, $f(\epsilon) < 0$ or $f(\epsilon) \approx 0$, because

$$f(\epsilon) = f\left((1-|\epsilon|\,)0 + |\epsilon| \cdot \frac{\epsilon}{|\epsilon|}\right) \leq (1-|\epsilon|\,)f(0) + |\epsilon| \cdot f\left(\frac{\epsilon}{|\epsilon|}\right) \approx 0;$$

2. $f(\epsilon) \gtrsim 0$, *i.e.*, $f(\epsilon) > 0$ or $f(\epsilon) \approx 0$, because

$$0 = \frac{1}{1+|\epsilon|}\epsilon + \frac{|\epsilon|}{1+|\epsilon|} \cdot \frac{-\epsilon}{|\epsilon|}$$

and so

$$0 \leq \frac{1}{1+|\epsilon|} f(\epsilon) + \frac{|\epsilon|}{1+|\epsilon|} f\left(\frac{-\epsilon}{|\epsilon|}\right) \Rightarrow f(\epsilon) \geq -|\epsilon| \cdot f\left(\frac{-\epsilon}{|\epsilon|}\right) \approx 0.$$

Therefore $f(\epsilon) \approx 0$. ■

We will now see the special case when E is a finite dimensional space. First we need the following result due to Michel Goze:

Theorem 2.2 *[Goz95] Let $M \in {}^*\mathbb{R}^n$ be an infinitesimal vector. Then there are nonzero infinitesimals $\epsilon_1, \ldots, \epsilon_k \in {}^*\mathbb{R}$ and standard vectors $V_1, \ldots, V_k \in \mathbb{R}^n$, for some $k \leq n$, such that*

$$M = \epsilon_1 V_1 + \epsilon_1 \epsilon_2 V_2 + \ldots + \epsilon_1 \epsilon_2 \ldots \epsilon_k V_k.$$

With this we can prove the following well-known theorem:

Theorem 2.3 *[Alm07b] Every convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.*

Proof. Again we assume that $x_0 = 0$ and $f(x_0) = 0$. Fix any $\epsilon \approx 0$ and write $\epsilon = \epsilon_1 V_1 + \epsilon_1 \epsilon_2 V_2 + \ldots + \epsilon_1 \epsilon_2 \ldots \epsilon_k V_k$ as above. We can also assume that all the infinitesimals ϵ_i are positive (replacing V_i by $-V_i$ if necessary).

1. $f(\epsilon) \lesssim 0$:

$$f(\epsilon) = f\left((1-\epsilon_1)0 + \epsilon_1(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k)\right)$$

$$\leq (1-\epsilon_1) f(0) + \epsilon_1 f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k).$$

It is enough to prove that $f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k)$ is finitely bounded from above:

$$f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k) \tag{2.1}$$

$$= f\left((1-\epsilon_2)V_1 + \epsilon_2(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_3 \ldots \epsilon_k V_k)\right)$$

$$\leq (1-\epsilon_2)f(V_1) + \epsilon_2 f(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_3 \ldots \epsilon_k V_k).$$

To see that $f(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_3 \ldots \epsilon_k V_k)$ is bounded above, we have

$$f(V_1 + V_2 + \epsilon_3 V_3 \ldots + \epsilon_3 \ldots \epsilon_k V_k)$$

$$= f\left((1-\epsilon_3)(V_1 + V_2) + \epsilon_3(V_1 + V_2 + V_3 + \epsilon_4 V_4 + \ldots. + \epsilon_4 \ldots \epsilon_k V_k)\right)$$

$$\leq (1-\epsilon_3)f(V_1 + V_2) + \epsilon_3 f(V_1 + V_2 + V_3 + \epsilon_4 V_4 + \ldots + \epsilon_4 \ldots \epsilon_k V_k).$$

Repeating this process we obtain

$$f(V_1 + V_2 + \ldots + \epsilon_k V_k) \leq (1-\epsilon_k)f(V_1 + V_2 + \ldots + V_{k-1}) + \epsilon_k f(V_1 + V_2 + \ldots + V_k)$$

which is bounded from above.

2. $f(\epsilon) \gtrsim 0$:

Since

$$0 = \frac{1}{1+\epsilon_1}\epsilon + \frac{\epsilon_1}{1+\epsilon_1} \cdot \frac{-\epsilon}{\epsilon_1}$$

we obtain

$$0 \leq \frac{1}{1+\epsilon_1} f(\epsilon) + \frac{\epsilon_1}{1+\epsilon_1} f\left(\frac{-\epsilon}{\epsilon_1}\right) \Rightarrow f(\epsilon) \geq -\epsilon_1 f\left(\frac{-\epsilon}{\epsilon_1}\right).$$

Note that

$$f\left(\frac{-\epsilon}{\epsilon_1}\right) = f(-V_1 - \epsilon_2 V_2 - \ldots - \epsilon_2 \ldots \epsilon_k V_k).$$

As in (2.1), replacing V_i by $W_i := -V_i$, we conclude that $f\left(\frac{-\epsilon}{\epsilon_1}\right)$ is bounded from above and hence $f(\epsilon) \gtrsim 0$.

$\blacksquare$

Theorem 2.4 *[Alm07b] Let $(E, |\cdot|)$ be a finite dimensional normed space, $(F, \mathcal{T})$ a Hausdorff linear topological space and $f : E \to F$ a function. If the image of every compact subset of E is compact in F and the image of every convex subset of E is convex in F, then f is continuous.*

Proof. Fix $x \in E$ and $y \in {}^*E$ with $y \approx x$. For every $n \in \mathbb{N}$, the closed ball $\overline{B}_{1/n}(x)$ is compact and convex, so $F_n := f\left(\overline{B}_{1/n}(x)\right)$ is also compact and convex. Besides, for each $n \in \mathbb{N}$,

$$x, y \in {}^*\overline{B}_{1/n}(x) \Rightarrow f(x), f(y) \in {}^*F_n \Rightarrow f(x), st(f(y)) \in F_n.$$

Since F_n is convex, for each $n \in \mathbb{N}$, there exists $x_n \in \overline{B}_{1/n}(x)$ with

$$f(x_n) = \frac{1}{n} f(x) + \left(1 - \frac{1}{n}\right) st(f(y)).$$

Since $\lim x_n = x$, the set $A := \{x\} \cup \{x_n | \, n \in \mathbb{N}\}$ is compact and so $f(A) = \{f(x_n) | n \in \mathbb{N}\}$ is also compact. Consequently, $f(x) = st(f(y))$ and f is continuous at x. ■

In particular, for real functions with real variable, we obtain a known result.

Theorem 2.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If the image of every compact subset of $\mathbb{R}$ is compact and the image of every connected subset of $\mathbb{R}$ is connected, then f is continuous.*

For a standard approach to this subject, see [Hal60], [McM70] and [GJSS04].

2.2 Connectedness and Compactness on Standard Sets

Nowadays there is no simple nonstandard characterization of connectedness and few work has been done in that direction. We refer [Let95] and [Rod01] for further reading.

In what follows, (X, d) will denote a metric space and $A \subseteq X$ a nonempty subset. Given two points $x, y \in {}^*A$, we define the set (possibly external)

$$\mathcal{P}_{x,y}^{*A} := \{u = (u_n)_{n=1,\ldots,N} \,|\, N \in {}^*\mathbb{N}, u_1 = x, u_N = y, u_n \in {}^*A$$

$$\text{and } u_n \approx u_{n+1}, \text{ for all } n \in \{1, \ldots, N-1\}\}.$$

The hyper-finite sequence $u = (u_n)_{n\in\{1,\ldots N\}}$ is called a **discrete infinitesimal path** (abbreviation **d.i.p.**) joining x to y in ${}^*A$. We define a binary relation on ${}^*A$ by $x \sim y$ if $\mathcal{P}^{*A}_{x,y}$ is nonempty; it is easy to prove that $\sim$ is an equivalence relation.

We will simply write $\mathcal{P}_{x,y}$ instead of $\mathcal{P}^{*A}_{x,y}$ whenever there is no danger of confusion.

Theorem 2.6 *[AN07] Let $f : X \to Y$ be a function. If f is continuous then for all subset $A \subseteq X$ for which*

$$\forall x, y \in {}^*A\, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^\sigma A, \text{ for all } n \tag{2.2}$$

the following condition is verified

$$\forall z, w \in {}^*f(A)\, \exists v \in \mathcal{P}_{z,w} \text{ with } v_n \in ns({}^*Y) \text{ and } st(v_n) \in {}^\sigma f(A), \text{ for all } n.$$

Proof. Let A be a set satisfying condition (2.2). Given z and w in ${}^*f(A)$, let $z = f(x)$ and $w = f(y)$, for some $x, y \in {}^*A$. Then there exists $u = (u_n)_{n=1,\ldots,N} \in \mathcal{P}_{x,y}$ with $u_n \in ns({}^*X)$ and $st(u_n) \in {}^\sigma A$ for all $n = 1, \ldots, N$. Define $v_n := f(u_n), n = 1, \ldots, N$. It is easy to see that $v = (v_n)$ satisfies the necessary conditions. ■

Theorem 2.7 *[AN07] The set A is connected if*

$$\forall x, y \in {}^\sigma A\, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^\sigma A, \text{ for all } n. \tag{2.3}$$

Proof. Assume that A is not connected. Hence A has a subset $B \notin \{\emptyset, A\}$ simultaneously relatively open and closed. Pick $x \in {}^\sigma B$, $y \in {}^\sigma(A - B)$ and $u = (u_n)_{n=1,\ldots,N} \in \mathcal{P}_{x,y}$ with $u_n \in ns({}^*X)$ and $st(u_n) \in {}^\sigma A$, for all n.

Define the internal set

$$K := \{n \in \{1, \ldots, N\} \mid u_n \in {}^*B\}.$$

Since K is nonempty ($1 \in K$), it has a maximum. Let $k := \max K$. Since $y \notin {}^*B$ then $k \neq N$. Besides this, $u_k \in {}^*B$ and $u_{k+1} \in {}^*(A - B)$. Since B and $A - B$ are both closed, $st(u_k) \in B$ and $st(u_{k+1}) \in A - B$.

As $u_k \approx u_{k+1}$ then $st(u_k) = st(u_{k+1}) \in {}^\sigma B \cap {}^\sigma(A - B)$, which ends the proof. ■

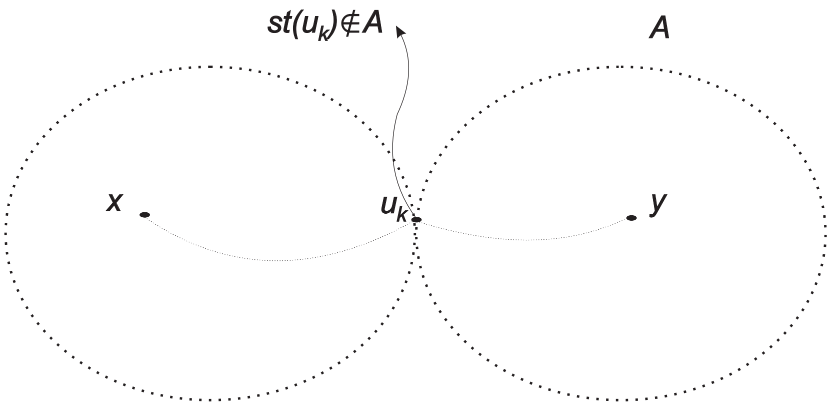

Figure 2.1

Remark 2.8 The previous condition is not enough to assert that A is path connected; simply take the set $\{(x, \sin(1/x)) \,|\, x > 0\} \cup (\{0\} \times [-1,1])$. However, if A is path connected then condition (2.3) is satisfied. Indeed, if we fix two points $x, y \in {}^{\sigma}A$ by hypothesis there exists a continuous path $\alpha : [0,1] \to A$ with $\alpha(0) = x$ and $\alpha(1) = y$. Take $N \in {}^{*}\mathbb{N}_{\infty}$ and define $u_n := \alpha(\frac{n}{N})$ for $n \in \{0, \ldots, N\}$. It is easy to verify that (u_n) verifies condition (2.3).

The converse of Theorem 2.7 is false in general. But there is a related result; first we will need the following theorem (the reader can see a proof of it in [New92]).

Theorem 2.9 *Let A be a connected set. Then for all open cover $E = \{E_z\}$ of A, if $E_{z_1}, E_{z_N} \in E$, with $N \in \mathbb{N}$, then there are $E_{z_2}, \ldots, E_{z_{N-1}} \in E$ with $E_{z_i} \cap E_{z_{i+1}} \neq \emptyset, i = 1, 2, \ldots, N-1$.*

Theorem 2.10 *[AN07] If A is a connected set then for all $x, y \in {}^{*}A$ holds $\mathcal{P}_{x,y} \neq \emptyset$.*

Proof. Fix $x, y \in {}^{\sigma}A$ and $\epsilon \in {}^{\sigma}\mathbb{R}^{+}$. Since $\{B_{\epsilon/2}(z) \,|\, z \in {}^{\sigma}A\}$ is an open cover of A and

$$B_{\epsilon/2}(x), B_{\epsilon/2}(y) \in \{B_{\epsilon/2}(z) \,|\, z \in {}^{\sigma}A\},$$

there are $u_2, \ldots, u_{N-1} \in {}^{\sigma}A$ with

$$B_{\epsilon/2}(u_i) \cap B_{\epsilon/2}(u_{i+1}) \neq \emptyset, \; i = 1, 2, \ldots, N-1$$

(here, $u_1 := x$ and $u_N := y$). So $d(u_i, u_{i+1}) < \epsilon$ for all i. In conclusion, the following sentence is true:

$$\forall x, y \in {}^{\sigma}A \, \forall \epsilon \in {}^{\sigma}\mathbb{R}^+ \, \exists N \in {}^{\sigma}\mathbb{N} \, \exists \{u_2, \dots, u_{N-1}\} \subset {}^{\sigma}A$$

$$\forall i \in \{1, \dots, N-1\} \quad d(u_i, u_{i+1}) < \epsilon.$$

Pick two points $x, y \in {}^*A$. By the Transfer Principle, this holds with $\epsilon \approx 0$. ■

Observe that we actually proved that, for all infinitesimal ϵ, there is $u \in \mathcal{P}_{x,y}$ satisfying $d(u_i, u_{i+1}) < \epsilon$.

Unfortunately, the *d.i.p.* need not to be nearstandard in A, as it is shown in the next example.

Let A be the subset of $\mathbb{R}^2$ defined by the condition (see Figure 2.2)

$$([0,1] \times \{0\}) \cup \left\{ \left(0, \frac{1}{n}\right) \mid n \in \mathbb{N} \right\} \cup \left\{ \left(\frac{1}{n}, y\right) \mid n \in \mathbb{N}, y \in [0,1] \right\}.$$

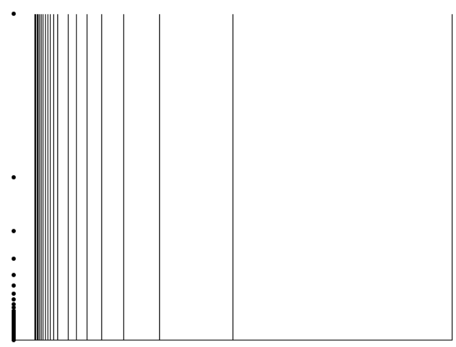

Figure 2.2

The set A is connected but there is no *d.i.p.* joining $(0,0)$ to $(0,1)$ nearstandard in the set.

Corollary 2.11 *[AN07] Let A be a compact set. Then A is connected if and only if*

$$\forall x, y \in {}^{\sigma}A \, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^{\sigma}A, \text{ for all } n.$$

Proof. It follows from Theorems 2.7 and 2.10 and the fact that, on compact sets, all points are nearstandard on the set. ■

In conclusion, we have now a nice characterization of connected compact sets.

Corollary 2.12 *[AN07] Let A be a non-empty set. Then A is connected and compact if and only if*

$$\forall x, y \in {}^*A\, \exists u \in \mathcal{P}_{x,y} \text{ with } u_n \in ns({}^*X) \text{ and } st(u_n) \in {}^\sigma A, \text{ for all } n. \tag{2.4}$$

Proof. We only need to prove that condition (2.4) implies the compactness condition. Fix any $x \in {}^*A$. By condition (2.4) there exists some $u \in \mathcal{P}_{x,x}$ nearstandard on A. So $u_1 = x \in ns({}^*X)$ and $st(x) \in {}^\sigma A$. ■

As we observed in page 3 we avoided local path-connectedness as well as using the monad of the set.

Chapter 3

A Nonstandard Approach to the Mean Value Theorem

The Mean Value Theorem is one of the most important theorems in Analysis; it has numerous formulations either integral or differential; we will work with the differential version in arbitrary normed spaces. The most important result in this chapter is the converse of the Mean Value Theorem, a generalization of a theorem presented in [TB97].

3.1 A Nonstandard Proof of the Mean Value Theorem

Let $(E, |\cdot|)$ and $(F, |\cdot|)$ be two normed spaces. As usual, we use the symbol $[x, y]$, where x and y are two vectors in E, to denote the elements of the closed line segment joining x with y. First we prove a nonstandard analogous of the Intermediate Value Theorem for internal functions.

Theorem 3.1 ***Intermediate Value Theorem*** *[AN06] Let $a, b \in {}^*\mathbb{R}$ and $f : [a, b] \to {}^*\mathbb{R}$ be an internal SU-continuous function with $f(a) < f(b)$. Then, for all $K \in {}^*\mathbb{R}$ with $f(a) < K < f(b)$, there exists $c \in [a, b]$ with $f(c) \approx K$.*

Proof. Fix $N \in {}^*\mathbb{N}_\infty$ with $(b-a)/N \approx 0$ and define

$$A := \left\{ j \in \{0, 1, \ldots, N\} \mid f\left(a + j\frac{b-a}{N}\right) < K \right\}.$$

The set A is nonempty since $0 \in A$. Let l be the maximum of A. Then $l \neq N$ and therefore we have

$$K \leq f\left(a + (l+1)\frac{b-a}{N}\right) \approx f\left(a + l\frac{b-a}{N}\right) < K$$

then

$$f\left(a + l\frac{b-a}{N}\right) \approx K.$$

■

Theorem 3.2 *[AN05b] Let U be an open convex subset of E and $f : U \to \mathbb{R}$ a C^1 function. Then, for all $x, y \in U$, there exists $c \in [x, y]$ with $f(x) - f(y) = Df_c(x - y)$.*

Proof. Fix an infinite $N \in {}^*\mathbb{N}_\infty$ and define $\delta := (y - x)/N \approx 0$. Then, for some infinitesimal numbers $\eta_1, \ldots, \eta_N$, the following holds:

$$\begin{aligned} f(x) - f(y) &= \sum_{n=1}^{N} [f(x + (n-1)\delta) - f(x + n\delta)] \\ &= \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x - y)}{N} + \frac{\sum_{n=1}^{N} \eta_n}{N}|x - y|. \end{aligned}$$

By the Transfer Principle

$$\left| \frac{\sum_{n=1}^{N} \eta_n}{N} \right| \leq \max\{|\eta_1|, ..., |\eta_N|\} \approx 0.$$

As a result

$$f(x) - f(y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x - y)}{N} \right). \tag{3.1}$$

Let $t_m, t_M \in \{0, ..., N-1\}$ be the hyper-integers satisfying

$$Df_{x+t_m\delta}(x - y) = \min_{t \in \{0,...,N-1\}} Df_{x+t\delta}(x - y)$$

and

$$Df_{x+t_M\delta}(x - y) = \max_{t \in \{0,...,N-1\}} Df_{x+t\delta}(x - y).$$

Then the following is verified:

$$Df_{x+t_m\delta}(x-y) \leq \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N} \leq Df_{x+t_M\delta}(x-y).$$

As $Df_{(\cdot)}(x-y)$ is a S-continuous function, the map

$$t \mapsto Df_{x+t\delta}(x-y),\ t \in [0, N-1]$$

is an internal SU-continuous function and so, by the Intermediate Value Theorem, there exists $k \in [x+t_m\delta, x+t_M\delta] \subseteq {}^*[x,y]$ with

$$Df_k(x-y) \approx \frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}.$$

Therefore, taking standard parts on the last equation and by (3.1), we obtain

$$f(x) - f(y) = Df_c(x-y)$$

where $c = st(k)$. ■

In the last theorem we proved that, for $N \approx \infty$ and $\delta = (y-x)/N$,

$$f(x) - f(y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}\right).$$

So, if $c \in [x,y]$ satisfies the condition

$$f(x) - f(y) = Df_c(x-y),$$

then

$$Df_c(x-y) = st\left(\frac{\sum_{n=1}^{N} Df_{x+(n-1)\delta}(x-y)}{N}\right).$$

In particular, if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a C^1 function, we get

$$f'(c)(x-y) = st\left(\frac{\sum_{n=1}^{N} f'(x+(n-1)\delta)(x-y)}{N}\right)$$

hence

$$f'(c) = st\left(\frac{f'(x) + f'(x+\delta) + f'(x+2\delta) + \ldots + f'(x+(N-1)\delta)}{N}\right),$$

i.e., c is the point in $[x,y]$ for which the derivative of f at c is the limit of the arithmetic *mean* of the derivatives of f at $x+(n-1)\delta$, $n = 1, \ldots, N$, as $N \to \infty$.

Analogously, we have

Theorem 3.3 *[AN05b] Let U be an open convex subset of E and $f : U \to F$ a C^1 function. Then, for all $x, y \in U$, there exists $c \in [x, y]$ with $|f(x) - f(y)| \leq |Df_c(x - y)|$.*

Proof. Since

$$|f(x) - f(y)| \leq st\left(\frac{\sum_{n=1}^{N} |Df_{x+(n-1)\delta}(x - y)|}{N}\right)$$

and taking t_m, t_M with

$$|Df_{x+t_m\delta}(x - y)| = \min_{t\in\{0,\dots,N-1\}} |Df_{x+t\delta}(x - y)|$$

and

$$|Df_{x+t_M\delta}(x - y)| = \max_{t\in\{0,\dots,N-1\}} |Df_{x+t\delta}(x - y)|,$$

we obtain the desired result. ∎

3.2 A Mean Value Theorem for Internal Functions

We now present the Mean Value Theorem for internal functions. Since the derivative function of an internal function is generally not unique, in the formula we must add an error. The proof will be omitted for it is similar to the proof of the same theorem for standard functions.

Theorem 3.4 *[AN05b] Let U be an open convex subset of E. If $f : {}^*U \to {}^*\mathbb{R}$ is an internal SU-differentiable function then*

$$\forall x, y \in ns({}^*U)\, \exists c \in [x, y] \quad f(x) - f(y) = Df_c(x - y) + |x - y|\eta,$$

for some $\eta \approx 0$.

More general, if $f : {}^*U \to {}^*F$ is an internal SU-differentiable function, then

$$\forall x, y \in ns({}^*U)\, \exists c \in [x, y] \quad |f(x) - f(y)| \leq |Df_c(x - y)| + |x - y|\eta,$$

with $\eta \approx 0$.

3.3 An Estimation for the Differential Mean Point

Fix $x \in U$ and assume that $y \in {}^*E$ is infinitely close to x. Where in the interval $[x, y]$ might c be located? We will begin by proving that, under some conditions, c approaches the midpoint of the segment $[x, y]$.

Let $f : U \subseteq E \to \mathbb{R}$ be a C^2 function, where U is an open convex set, and fix $x \in U$. Then, for all $y \in U$, by the Mean Value Theorem, we can ensure the existence of $c \in [x, y]$ with $f(x) - f(y) = Df_c(x - y)$. By transfer, if $y \in {}^*U$ with $y \approx x$, there still exists such $c \in [x, y]$. We give a generalization of a result due to Jacobson, presented in [Jac82]:

Theorem 3.5 *[AN05b] Under the previous assumptions and, if*

$$D^2 f_x \left(\frac{x-y}{|x-y|} \right)^{(2)} \not\approx 0,$$

then

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{2}.$$

Proof. Since f is twice continuously differentiable, we have:

- $f(x) - f(y) = Df_x(x - y) + 1/2 D^2 f_x(x - y)^{(2)} + |x - y|^2 \eta$, for some $\eta \approx 0$;
- $Df_c(x-y) = Df_x(x-y) + D^2 f_x(x-y, c-x) + |x-c| \cdot \theta(x-y)$, where $\theta \in Lin({}^*E, {}^*\mathbb{R})$ is an operator such that $\theta(fin({}^*E)) \subseteq inf({}^*\mathbb{R})$ (see ([Str78]);

and also the equality

- $f(x) - f(y) = Df_c(x - y)$.

Therefore

$$D^2 f_x(x-y, c-x) + |x-c| \cdot \theta(x-y) = \frac{1}{2} D^2 f_x(x-y)^{(2)} + |x-y|^2 \eta \Leftrightarrow$$

$$|x-y| \cdot |x-c| \left[D^2 f_x \left(\frac{x-y}{|x-y|}, \frac{c-x}{|x-c|} \right) + \theta \left(\frac{x-y}{|x-y|} \right) \right] = |x-y|^2 \left[\frac{1}{2} D^2 f_x \left(\frac{x-y}{|x-y|} \right)^{(2)} + \eta \right]$$

which implies

$$\frac{|x-c|}{|x-y|} = \frac{1}{2}\frac{\left|D^2 f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} + 2\eta\right|}{\left|-D^2 f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} + \theta\left(\frac{x-y}{|x-y|}\right)\right|} \approx \frac{1}{2}.$$

∎

In the paper *A Note on the Mean Value Theorem for Integrals*, Zhang Bao-lin extends the result of Jacobson (see [Bl97]). Next we generalize his work for arbitrary normed spaces.

Theorem 3.6 *[AN05b] Let $f : U \subseteq E \to \mathbb{R}$ be a C^3 function, where U is an open convex set. If*

1. $x \in {}^{\sigma}U$, $y \in {}^{*}U$ *with* $y \approx x$;
2. $c \in [x, y]$ *with* $f(x) - f(y) = Df_c(x - y)$;
3. $D^2 f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} = 0$ *and* $D^3 f_x\left(\frac{x-y}{|x-y|}\right)^{(3)} \not\approx 0$,

then

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{\sqrt{3}}.$$

Proof. Taking the Taylor's expansions:

- $f(x) - f(y) = Df_x(x-y) + 1/2D^2 f_x(x-y)^{(2)} + 1/6D^3 f_x(x-y)^{(3)} + |x-y|^3\eta$;
- $Df_c(x-y) = Df_x(x-y) + D^2 f_x(x-y, c-x) + 1/2D^3 f_x(x-y, c-x, c-x) + |x-c|^2 \cdot \theta(x-y)$;

roughly as before

$$\left(\frac{|x-c|}{|x-y|}\right)^2 = \frac{\left|\frac{1}{2|x-y|}D^2 f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} + \frac{1}{6}D^3 f_x\left(\frac{x-y}{|x-y|}\right)^{(3)} + \eta\right|}{\left|-\frac{1}{|x-c|}D^2 f_x\left(\frac{x-y}{|x-y|}\right)^{(2)} + \frac{1}{2}D^3 f_x\left(\frac{x-y}{|x-y|}\right)^{(3)} + \theta\left(\frac{x-y}{|x-y|}\right)\right|} \approx \frac{1}{3}.$$

∎

Iterating this procedure, the following results:

Theorem 3.7 *[AN05b] Let $f : U \subseteq E \to \mathbb{R}$ be a C^{k+1} function, where U is an open convex set. If*

1. $x \in {}^{\sigma}U$, $y \in {}^{*}U$ *with* $y \approx x$;
2. $c \in [x, y]$ *with* $f(x) - f(y) = Df_c(x - y)$;
3. $D^j f_x \left(\frac{x-y}{|x-y|}\right)^{(j)} = 0$, *for* $j = 2, \ldots, k$ *and* $D^{k+1} f_x \left(\frac{x-y}{|x-y|}\right)^{(k+1)} \not\approx 0$,

then

$$\frac{|x-c|}{|x-y|} \approx \frac{1}{\sqrt[k]{k+1}}.$$

Proof. Just observe that the Taylor's expansions are now

- $$\begin{aligned} f(x) - f(y) = \;& Df_x(x-y) + 1/2D^2 f_x(x-y)^{(2)} + 1/3!D^3 f_x(x-y)^{(3)} + \ldots \\ & +1/(k+1)!D^{k+1} f_x(x-y)^{(k+1)} + |x-y|^{k+1}\eta; \end{aligned}$$
- $$\begin{aligned} Df_c(x-y) = \;& Df_x(x-y) + D^2 f_x(x-y, c-x) + 1/2D^3 f_x(x-y, c-x, c-x) + \ldots \\ & +1/k!D^{k+1} f_x(x-y, c-x, c-x, \ldots, c-x) + |x-c|^k \cdot \theta(x-y). \end{aligned}$$

∎

3.4 A Converse of the Mean Value Theorem

Given a differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and a point $c \in I$, are there reals $a, b \in I$ such that $c \in]a, b[$ and $f(b) - f(a) = f'(c)(b - a)$? A simple example shows that the converse of the Mean Value Theorem may fail. For the function $f(x) = x^3, x \in [-1, 1]$ and $c = 0$, we have $f'(0) = 0$ yet f is $1 - 1$. In [TB97] is presented a theorem that establish sufficient conditions for the converse to hold.

Theorem 3.8 *[TB97] Let f be a continuous function in $[a,b]$ and differentiable in $]a,b[$ and let $c \in]a,b[$. Then*

1. ***Weak Form:*** *If $f'(c) \neq sup\{f'(x) \,|\, x \in]a,b[\}$ and $f'(c) \neq inf\{f'(x) \,|\, x \in]a,b[\}$, then there are $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$.*
2. ***Strong Form:*** *If $f'(c) \neq sup\{f'(x) \,|\, x \in]a,b[\}$, $f'(c) \neq inf\{f'(x) \,|\, x \in]a,b[\}$ and c is not an accumulation point of the set $\{x \in]a,b[\,|\, f'(x) = f'(c)\}$, then there are $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 < c < a_1$.*

We now extend their result for functions whose domain is a subset of a normed space.

Theorem 3.9 *[AN05b] Let $f : U \subseteq E \to \mathbb{R}$ be a C^1 function, where U is an open set. Given $c \in U$, suppose that there exists $v \in E$ such that:*

1. *$\{c + tv \,|\, -1 \leq t \leq 1\} \subseteq U$;*
2. *$\forall\, 0 < t \leq 1 \quad Df_{c-tv}(v) \leq Df_c(v) \leq Df_{c+tv}(v)$.*

Then there are $a, b \in U$ satisfying $f(b) - f(a) = Df_c(b-a)$. Furthermore, $c \in [a,b]$.

Proof. Let us fix $\epsilon, k \in \mathbb{R}$ with $0 < \epsilon < k < 1$ and define

$$L := \frac{f(c + (\epsilon - k)v) - f(c + kv)}{\epsilon - 2k} \in \mathbb{R}.$$

Then $-1 < -k < \epsilon - k < 0$ and so the constant L is well defined. We will divide the proof into two different cases:

First case: $L \geq Df_c(v)$:

Define the function

$$g(t) := \frac{f(c + (\epsilon - k)v) - f(c + tkv)}{\epsilon - k - tk},\ t \in [0,1].$$

The function is well defined since

$$\epsilon - k - tk = 0 \Leftrightarrow t = \epsilon/k - 1$$

and $-1 < \epsilon/k - 1 < 0$. Furthermore, g is continuous and $g(1) = L \geq Df_c(v)$. By the Mean Value Theorem, for each $t \in [0,1]$, there exists $d(t) \in [c + (\epsilon - k)v, c + tkv]$ with

$$f(c + (\epsilon - k)v) - f(c + tkv) = Df_{d(t)}((\epsilon - k - tk)v).$$

So $g(0) = Df_{d(0)}(v)$ for some $d(0) \in [c + (\epsilon - k)v, c]$. If $d(0) = c$, then

$$f(c + (\epsilon - k)v) - f(c) = Df_c((\epsilon - k)v)$$

and the theorem is proved for $a = c$ and $b = c + (\epsilon - k)v$. If that is not the case, *i.e.*, $d(0) \in [c + (\epsilon - k)v, c[$, by the hypothesis of the theorem, we have $g(0) \leq Df_c(v)$. Using the Intermediate Value Theorem, we guarantee the existence of $t \in [0,1]$ satisfying the condition $g(t) = Df_c(v)$, *i.e.*,

$$f(c + (\epsilon - k)v) - f(c + tkv) = Df_c((\epsilon - k - tk)v).$$

In this case we take $a = c + tkv$ and $b = c + (\epsilon - k)v$.

Second case: $L < Df_c(v)$:

Analogously, we begin by defining the continuous function

$$h(t) := \frac{f(c + (\epsilon - tk)v) - f(c + kv)}{\epsilon - k - tk},\ t \in [\epsilon/k, 1].$$

For each $t \in [\epsilon/k, 1]$, there exists $d(t) \in [c + (\epsilon - tk)v, c + kv]$ satisfying

$$f(c + (\epsilon - tk)v) - f(c + kv) = Df_{d(t)}((\epsilon - k - tk)v).$$

Then $h(1) < Df_c(v)$ and $h(\epsilon/k) = Df_{d(\epsilon/k)}(v)$, for some $d(\epsilon/k) \in [c, c + kv]$. If $d(\epsilon/k) = c$, then

$$f(c + kv) - f(c) = Df_c(kv)$$

and we choose $a = c$ and $b = c + kv$. If $d(\epsilon/k) \in]c, c + kv]$, then $h(\epsilon/k) \geq Df_c(v)$. Again, by the Intermediate Value Theorem, there exists some $t \in [\epsilon/k, 1[$ with $h(t) = Df_c(v)$, *i.e.*,

$$f(c + (\epsilon - tk)v) - f(c + kv) = Df_c((\epsilon - k - tk)v)$$

and in this case we take $a = c + kv$ and $b = c + (\epsilon - tk)v$. ∎

It is obvious that the last theorem holds if we replace condition *2.* by

$$\forall\, 0 < t \leq 1 \qquad Df_{c+tv}(v) \leq Df_c(v) \leq Df_{c-tv}(v)$$

Finally, for real functions with real variable, we have

Theorem 3.10 *[AN05b] Let f be a continuous function in $[a,b]$ and differentiable in $]a,b[$ and let $c \in]a,b[$. Suppose that there exists $k_0 > 0$ with $]c-k_0, c+k_0[\subseteq]a,b[$ and for all $k \in]0,k_0[$,*

1. ***Weak Form:** if $f'(c-k) \leq f'(c) \leq f'(c+k)$ then there exist $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 \leq c \leq a_1$.*

2. ***Strong Form:** if $f'(c-k) < f'(c) < f'(c+k)$ then there exist $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 < c < a_1$.*

Proof. Let us begin by proving the Weak Form of the theorem. Let $0 < \epsilon < k < k_0$ be fixed reals and define

$$L := \frac{f(c+\epsilon-k) - f(c+k)}{\epsilon - 2k}.$$

First case: $L \geq f'(c)$:

The function

$$g(t) := \frac{f(c+\epsilon-k) - f(c+tk)}{\epsilon - k - tk},\ t \in [0,1]$$

is continuous and $g(1) = L \geq f'(c)$.

By the Mean Value Theorem, for each $t \in [0,1]$, there exists $d(t) \in]c+\epsilon-k, c+tk[$ with $g(t) = f'(d(t))$. Then $g(0) = f'(d(0))$ with $d(0) \in]c+\epsilon-k, c[$. Therefore $g(0) \leq f'(c)$. By the Intermediate Value Theorem, there exists $t \in [0,1]$ with

$$\frac{f(c+\epsilon-k) - f(c+tk)}{\epsilon - k - tk} = f'(c)$$

and we prove the theorem with $a_1 = c + tk$ and $b_1 = c + \epsilon - k$.

Second case: $L < f'(c)$:

Just make the necessary adjustments to the previous case and the proof of Theorem 3.9.

To prove the Strong Form, simply note that in the first case we obtain $g(0) < f'(c)$, so there exists $t \in]0,1]$ with

$$\frac{f(c+\epsilon-k) - f(c+tk)}{\epsilon - k - tk} = f'(c)$$

Besides this, $c+\epsilon-k < c < c+tk$. The second case is analogous. This ends the proof. ■

Let us notice that none of the Theorems 3.8 and 3.10 is stronger than the other. For example, the function $f(x) = 1, x \in [-1,1]$ satisfies the hypothesis of Theorem 3.10 (the Weak Form) but none of the Theorem 3.8. For the converse, take $f(x) = x^4 + x^3, x \in [-1,1]$ and $c = 0$.

Furthermore, in the Weak Form of the last theorem, we can not state that $b_1 < c < a_1$. For instance, let $f : [-1,1] \to \mathbb{R}$ be the function given by $f(x) = -x^3$ if $x < 0$ and $f(x) = 0$ elsewhere. For $c = 0$ and $k \in]0,1[$ we have $f'(-k) \leq f'(0) \leq f'(k)$ but there are not two reals a_1, b_1 such that $f'(0) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 < 0 < a_1$.

Theorem 3.11 *[AN05b] Let $f : [a,b] \to \mathbb{R}$ be a continuous function in $[a,b]$ and differentiable in $]a,b[$ and let $c \in]a,b[$. Suppose that there exists $k_0 > 0$ with $]c-k_0, c+k_0[\subseteq]a,b[$ and for all $k \in]0,k_0[$,*

1. ***Weak Form:*** *if $f'(c+k) \leq f'(c) \leq f'(c-k)$ then there exist $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 \leq c \leq a_1$.*

2. ***Strong Form:*** *if $f'(c+k) < f'(c) < f'(c-k)$ then there exist $a_1, b_1 \in]a,b[$ such that $f'(c) = (f(b_1) - f(a_1))/(b_1 - a_1)$ and $b_1 < c < a_1$.*

Chapter 4

mu-differentiability of an Internal Function

The aim of this chapter is to introduce a new kind of differentiability, the mu-differentiability. Usually, in the literature on analysis in nonstandard terms, S-differentiability is prevailing. As we will see, mu-differentiability has some advantages when compared to SU-differentiability, namely when we deal with perturbations of classical functions.

4.1 The Definition

In 1992, M. Reeken presented a new type of differentiability, the *macroscopic differentiability* (m-differentiability for short):

Definition 4.1 *Let* $(E, |\cdot|)$ *and* $(F, |\cdot|)$ *be normed spaces,* U *be an open subset of* E *and* $f : {}^*U \to {}^*F$ *be an internal function. We say that* f *is* ***m-differentiable*** *at* $a \in {}^\sigma U$ *if*

1. *there exist* $0 \approx \delta_a \in {}^*\mathbb{R}^+$ *and a finite linear operator* $Df_a \in {}^*L(E, F)$ *such that, for all* $x \in {}^*U$ *with* $\delta_a < |x - a| \approx 0$*, there is some* $\eta \approx 0$ *such that*

$$f(x) - f(a) = Df_a(x - a) + |x - a|\eta$$

2. $f(ns({}^*U)) \subseteq ns({}^*F)$.

The function f is called m-differentiable if it is m-differentiable at all $a \in {}^\sigma U$.

Since $f(ns({}^*U)) \subseteq ns({}^*F)$, it makes sense to define the standard function

$$\begin{array}{rrcl} st(f): & {}^\sigma U & \rightarrow & {}^\sigma F \\ & x & \mapsto & st(f(x)) \end{array}$$

If g is a standard differentiable function and $\sup\limits_{x\in {}^*U} |f(x)-g(x)| \approx 0$, then f is m-differentiable. In fact, it can be proved that

Theorem 4.2 *[Sch97] If E and F are two standard finite dimensional normed spaces, K a standard compact subset of E and $f : {}^*K \rightarrow {}^*F$ an internal function, then the following statements are equivalent:*

1. *f is S-continuous and m-differentiable;*

2. *There exists a differentiable standard function $g : K \rightarrow F$ with*

$$\sup_{x\in {}^*K} |f(x) - g(x)| \approx 0.$$

This result played a very important role in the characterization of a nonstandard manifold concept presented in [Sch97]. Under some conditions, the internal transition functions φ_{ij} are S-continuous, m-differentiable with S-continuous m-derivative if and only if there exist standard C^1 transition functions infinitely close to φ_{ij}.

In this work we extend the last result for m-uniformly differentiable functions and study other properties of this differentiability. First we introduce the notion of mu-differentiability (short for m-uniformly differentiability).

Definition 4.3 *Let $f : {}^*U \to {}^*F$ be an internal function. We say that f is* ***mu-differentiable*** *if*

1. *for each $a \in {}^\sigma U$ there exists a positive infinitesimal δ_a such that, for all $x \in \mu(a)$, there exists a finite linear operator $Df_x \in {}^*L(E, F)$ for which holds*

$$\forall y \in \mu(a) \quad |x - y| > \delta_a \Rightarrow f(x) - f(y) = Df_x(x - y) + |x - y|\eta$$

for some $\eta \approx 0$.

2. *$f(ns({}^*U)) \subseteq ns({}^*F)$.*

Since $a \in \mu(a)$, every mu-differentiable function is m-differentiable.

For example, let

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \epsilon & \text{if } x = 0 \end{cases}$$

where ϵ is a positive infinitesimal number. Then f is mu-differentiable and $f'(x) = 0$ for every $x \in ns({}^*\mathbb{R})$. In fact, let $a = 0$ (when $0 \neq a \in {}^\sigma\mathbb{R}$ it is obvious) and let $x \approx y \approx 0$ with $|x - y| > \delta_0 := \sqrt{\epsilon}$. Then

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq \frac{\epsilon}{\sqrt{\epsilon}} \approx 0.$$

Observe that f is not S-differentiable (nor SU-differentiable) since

$$\frac{f(\epsilon^2) - f(0)}{\epsilon^2 - 0} = -\frac{\epsilon}{\epsilon^2}$$

is infinite.

In the next example the choice of δ_a is independent of the point a fixed.

The function $f(x) = [x]\epsilon, x \in {}^*\mathbb{R}$, where ϵ is any positive infinitesimal and $[x]$ is the biggest integer less than or equal to x, is mu-differentiable and $f'(x) = 0$, for every $x \in ns({}^*\mathbb{R})$. In fact, let $a \in {}^\sigma\mathbb{R}$ be a real and choose a positive infinitesimal δ such that ϵ/δ is still infinitesimal (for example, $\delta = \sqrt{\epsilon}$). If $x, y \approx a$ with $|x - y| > \delta$ then

1. if $a \notin \mathbb{Z}$ then $\dfrac{f(x) - f(y)}{x - y} = 0$;

2. if $a \in \mathbb{Z}$ and $x, y \geq a$ or $x, y < a$ then $\dfrac{f(x) - f(y)}{x - y} = 0$;
3. in the other cases, $\left|\dfrac{f(x) - f(y)}{x - y}\right| \leq \dfrac{\epsilon}{\delta} \approx 0$.

It is obvious that

Theorem 4.4 *[AN06] Let f and g be two mu-differentiable functions and $k \in ns({}^*\mathbb{R})$. Then $f + g$ and kf are mu-differentiable.*

Theorem 4.5 *[AN06] If the function $f : {}^*U \to {}^*F$ is mu-differentiable then*

$$\forall x, y \in ns({}^*U) \quad x \approx y \Rightarrow f(x) \approx f(y),$$

i.e., the function is S-continuous.

Proof. Let us fix $x, y \in ns({}^*U)$ with $x \approx y$ and let $a := st(x)$. Since $x, y \in \mu(a)$, there exist two finite linear operators $Df_x, Df_y \in {}^*L(E, F)$ such that, for all $z \in \mu(a)$

- $|x - z| > \delta_a \Rightarrow f(x) - f(z) = Df_x(x - z) + |x - z|\eta_1$,
- $|y - z| > \delta_a \Rightarrow f(y) - f(z) = Df_y(y - z) + |y - z|\eta_2$,

with $\eta_1 \approx \eta_2 \approx 0$. Choose any $z \in \mu(a)$ with $\min\{|x - z|, |y - z|\} > \delta_a$. Then

$$f(x) - f(z) \approx 0 \approx f(y) - f(z)$$

which concludes the proof. ■

Remark 4.6 m-differentiability of a function does not implies S-continuity. Let

$$\begin{array}{rccc} f : & {}^*]-1,1[& \longrightarrow & {}^*\mathbb{R} \\ & x & \longmapsto & \begin{cases} 0 & \text{if} \quad x \neq \epsilon \\ 1 & \text{if} \quad x = \epsilon \end{cases} \end{array}$$

where ϵ is a positive infinitesimal number. Then f is m-differentiable at $x = 0$ (take $\delta_0 = \epsilon$) but it is not S-continuous.

The next theorem shows that for mu-differentiable functions we have continuity for the derivative function $x \mapsto Df_x$:

Theorem 4.7 *[AN06] Let f be a mu-differentiable function, $x, y \in ns(^*U)$ with $x \approx y$. Then for all $d \in {}^*E$ with $|d| = 1$, $Df_x(d) \approx Df_y(d)$.*

Proof. Let $a = st(x)$ and $d \in {}^*E$ with $|d| = 1$. We will divide the proof in two cases. The first part of our proof is similar to Stroyan's ([Str78]).

First Case: $|x - y| > \delta_a$

Let $\epsilon := \sqrt{|x-y|}$ and $z := \epsilon d + x = \epsilon \left(d + \frac{x-y}{\epsilon}\right) + y$. Since

1. $0 \approx |x - y| > \delta_a$;
2. $0 \approx |z - x| = \epsilon > \delta_a$;
3. $0 \approx |z - y| \geq \epsilon(1 - \epsilon) > \delta_a$;

the following holds:

1. $f(x) - f(y) = Df_y(x - y) + \epsilon\eta_1$, $\eta_1 \approx 0$;
2. $f(z) - f(x) = \epsilon Df_x(d) + \epsilon\eta_2$, $\eta_2 \approx 0$;
3. $f(z) - f(y) = \epsilon Df_y(d) + Df_y(x - y) + \epsilon\eta_3$, $\eta_3 \approx 0$.

So we conclude that

$$f(x) - f(y) = \epsilon(Df_y(d) - Df_x(d)) + Df_y(x - y) + \epsilon\eta = Df_y(x - y) + \epsilon\eta_1$$

(for some infinitesimal η) and so $Df_x(d) \approx Df_y(d)$.

Second Case: $|x - y| \leq \delta_a$

Let $w \in {}^*U$ be such that

$$0 \approx |x - w| > \delta_a \quad \& \quad 0 \approx |y - w| > \delta_a$$

Then for all $d \in {}^*E$ with $|d| = 1$

$$Df_x(d) \approx Df_w(d) \approx Df_y(d).$$

■

We now present the main result of this chapter. It extends Theorem 4.2 for mu-differentiable functions. As one might expect, in this case, the internal function is infinitely close to a C^1 standard function.

Theorem 4.8 *[AN06] Let $f : {}^*U \to {}^*F$ be an internal function. Then:*

1. *If F is a finite dimensional space and f is a mu-differentiable function, then $st(f) : U \to F$ is a C^1 function and $Dst(f)_a = st(Df_a)$ for $a \in {}^\sigma U$. Furthermore, if E is also finite dimensional then*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - st(f)(x)| \leq \eta_0.$$

2. *If there exists a C^1 standard function $g : U \to F$ with*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - g(x)| \leq \eta_0,$$

then f is mu-differentiable. Moreover, $g = st(f)$.

Proof.

1. Suppose that F is a finite dimensional normed space and f is mu-differentiable. We will begin by proving that $st(f)$ is differentiable at $a \in {}^\sigma U$, *i.e.*, there exists a finite linear operator L_a such that

$$\forall \eta \in {}^\sigma\mathbb{R}^+\, \exists \epsilon \in {}^\sigma\mathbb{R}^+\, \forall h \in {}^\sigma E \quad 0 < |h| < \epsilon \Rightarrow \frac{|st(f)(a+h) - st(f)(a) - L_a(h)|}{|h|} < \eta.$$

Fix $\eta \in {}^{\sigma}\mathbb{R}^{+}$ and let

$$A := \Big\{\epsilon \in {}^{*}\mathbb{R}^{+} \mid \epsilon \leq \delta_a \vee [\forall h \in {}^{*}E$$

$$\delta_a < |h| < \epsilon \Rightarrow \frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} < \frac{\eta}{2}\Big]\Big\}.$$

Since A is an internal set and contains all positive infinitesimal numbers, by the Spillover Principle there exists $\epsilon \in {}^{\sigma}\mathbb{R}^{+}$ such that $\epsilon \in A$. Choose now $h \in {}^{\sigma}E$ with $0 < |h| < \epsilon$. As h is standard, $\delta_a < |h| < \epsilon$ and so

$$\frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} < \frac{\eta}{2}.$$

Taking standard parts one gets

$$\frac{|st(f)(a+h) - st(f)(a) - L_a(h)|}{|h|} < \eta$$

where $L_a := st(Df_a)$. So $st(f)$ is differentiable and $Dst(f)_a = st(Df_a)$ for $a \in {}^{\sigma}U$.

Next we will prove that the function $x \mapsto Dst(f)_x$ is continuous, *i.e.*,

$$\forall a \in {}^{\sigma}U\, \forall \eta \in {}^{\sigma}\mathbb{R}^{+}\, \exists \epsilon \in {}^{\sigma}\mathbb{R}^{+}\, \forall x \in {}^{\sigma}U\, \forall d \in {}^{\sigma}E$$

$$[|x-a| < \epsilon \wedge |d| = 1] \Rightarrow |Dst(f)_x(d) - Dst(f)_a(d)| < \eta.$$

Choose any $a \in {}^{\sigma}U$ and $\eta \in {}^{\sigma}\mathbb{R}^{+}$ and let

$$B := \Big\{\epsilon \in {}^{*}\mathbb{R}^{+} \mid \forall x \in {}^{*}U\, \forall d \in {}^{*}E$$

$$[|x-a| < \epsilon \wedge |d| = 1] \Rightarrow |Df_x(d) - Df_a(d)| < \frac{\eta}{2}\Big\}.$$

Again the internal set B contains all positive infinitesimals. In fact, if $0 < \epsilon \approx 0$, for any $x \in {}^{*}U$ and $d \in {}^{*}E$ with $|d| = 1$ and $|x-a| < \epsilon$, by Theorem 4.7, one has $Df_x(d) \approx Df_a(d)$ and so

$$|Df_x(d) - Df_a(d)| < \frac{\eta}{2}.$$

So B must contain a positive standard ϵ. Choose now $x \in {}^{\sigma}U$ and $d \in {}^{\sigma}E$ satisfying $|d| = 1$ and $|x-a| < \epsilon$; hence

$$|Df_x(d) - Df_a(d)| < \frac{\eta}{2},$$

which implies

$$|Dst(f)_x(d) - Dst(f)_a(d)| < \eta,$$

proving that $st(f)$ is a C^1 function.

Assume now that E is finite dimensional. Observe that for $a \in {}^\sigma U$ and $x \approx a$, we have

$$f(x) - st(f)(x) \approx f(a) - st(f)(a) = f(a) - st(f(a)) \approx 0.$$

Moreover, for every $a \in {}^\sigma U$, we can choose $n \in {}^\sigma\mathbb{N}$ with $B_{2/n}(a) \subseteq U$. So, if we define K as being the closed ball $\overline{B}_{1/n}(a)$, we have

$$a \in K \subseteq U.$$

Let

$$\eta_0 := \sup_{y \in {}^*K} |f(y) - st(f)(y)|.$$

It is easy to verify that $\eta_0 \approx 0$, which ends the proof of *1.*

2. Let $g \in C^1(U, F)$. Fix any $a \in {}^\sigma U$ and let $\delta_a := \sqrt{\eta_0}$. Choose any $x, y \in \mu(a)$ with $\delta_a < |x - y|$. Since g is continuously differentiable, there exists a finite linear operator Dg_x which satisfies the condition

$$g(x) - g(y) = Dg_x(x - y) + |x - y|\eta$$

for some $\eta \approx 0$.

For $\epsilon_1 := g(x) - f(x)$ and $\epsilon_2 := g(y) - f(y)$, it is true that $\max\{|\epsilon_1|, |\epsilon_2|\} \leq \eta_0$ and

$$f(x) - f(y) = Dg_x(x - y) + |x - y|\eta + \epsilon_2 - \epsilon_1.$$

Furthermore, we also have

$$\frac{|\epsilon_1 - \epsilon_2|}{|x - y|} \leq \frac{|\epsilon_1| + |\epsilon_2|}{|x - y|} \leq \frac{2\eta_0}{\sqrt{\eta_0}} \approx 0.$$

To see that $g = st(f)$, note that both are standard functions and for every $a \in {}^\sigma U$, $g(a) = st(f)(a)$.

$\blacksquare$

Remark 4.9 The previous theorem is false if we replace mu-differentiability by SU-differentiability. Of course *1* still holds since SU-differentiability is a stronger condition, but *2* may fail. For example, suppose $g(x) = 0, x \in \mathbb{R}$ and $f(x) = 0$, if $x \in {}^*\mathbb{R}\backslash\{0\}$ and $f(0) = \epsilon$ with $0 \neq \epsilon \in \mu(0)$. Then g is a standard C^1 function infinitely close to f but f is not SU-differentiable.

It is easy to prove that

Corollary 4.10 *For a standard function $f : U \to F$, the following conditions are equivalent:*

1. *f is of class C^1;*
2. *f is mu-differentiable.*

Note that, for a mu-differentiable function $f : {}^*U \to {}^*F$, we can define a new function

$$\begin{array}{rrcl} L_{(\cdot)}: & ns({}^*U) & \to & {}^*L(E,F) \\ & x & \mapsto & Df_x \end{array}$$

By the Comprehension Principle, there exists an internal function $Df_{(\cdot)} : {}^*U \to {}^*L(E,F)$ such that $Df|_{ns({}^*U)} = L$. Since $L(E,F)$ is still a standard normed space, we are able to define higher-order derivatives. We say that f is twice mu-differentiable provided f and $Df_{(\cdot)}$ are both mu-differentiable.

Recursively, f is k-times mu-differentiable provided f, $Df_{(\cdot)}$, ...,$D^{k-1}f_{(\cdot)}$ are all mu-differentiable.

Theorem 4.11 *[AN06] Let $f : {}^*U \to {}^*F$ be an internal function. Then:*

1. *If F is a finite dimensional space and f is k-times mu-differentiable, then $st(f) : U \to F$ is a C^k function and for each $a \in {}^\sigma U$, $D^j st(f)_a = st(D^j f_a)$ for $j = 1, 2, \ldots, k$. Furthermore, if E is also finite dimensional,*

$$\forall a \in {}^\sigma U \, \exists \eta_0 \approx 0 \, \forall x \approx a \quad |f(x) - st(f)(x)| \leq \eta_0$$

and

$$\forall j \in \{1, 2, \ldots, k-1\} \, \forall a \in {}^\sigma U \, \exists \eta_j \approx 0 \, \forall x \approx a \quad |D^j f_x - D^j st(f)_x| \leq \eta_j.$$

2. *If there exists a C^k standard function $g : U \to F$ with*

$$\forall a \in {}^\sigma U\, \exists \eta_0 \approx 0\, \forall x \approx a \quad |f(x) - g(x)| \leq \eta_0$$

and

$$\forall j \in \{1, 2, \ldots, k-1\}\, \forall a \in {}^\sigma U\, \exists \eta_j \approx 0\, \forall x \approx a \quad |D^j f_x - D^j g_x| \leq \eta_j$$

then f is k-times mu-differentiable. Moreover, $g = st(f)$.

Proof. The proof is by induction on k as follows:

For $k = 1$: it was proven in Theorem 4.8 that *1* and *2* hold.

For $k \Rightarrow k+1$:

We will begin by proving that *1* holds. Assume then that f is $(k+1)$-times mu-differentiable. By hypothesis of induction, $st(f)$ is of class C^k and satisfies the other conditions of *1*. Since

$$\begin{array}{rccc} D^k f_{(\cdot)}: & {}^*U & \to & {}^*L^k(E,F) \\ & x & \mapsto & D^k f_x \end{array}$$

is still mu-differentiable, its standard part

$$\begin{array}{rccc} st(D^k f_{(\cdot)}): & {}^\sigma U & \to & {}^\sigma L^k(E,F) \\ & x & \mapsto & st(D^k f_x) \end{array}$$

is of class C^1 and, for every $a \in {}^\sigma U$, $Dst(D^k f_a) = st(D(D^k f_a))$. But since, when a is standard, $st(D^k f_a) = D^k st(f)_a$,

- $D^k st(f)_{(\cdot)}$ is also of class C^1 and so $st(f)$ is of class C^{k+1};
- $D^{k+1} st(f)_a = st(D^{k+1} f_a)$.

Furthermore, for $a \in {}^\sigma U$ and $x \approx a$,

$$D^k f_x \approx D^k f_a \approx D^k st(f)_a \approx D^k st(f)_x.$$

Similarly as in the proof of Theorem 4.8, we can prove that there exists an infinitesimal number η_k for which holds

$$|D^k f_x - D^k st(f)_x| \leq \eta_k$$

whenever $x \approx a$ and E is a finite dimensional normed space, which ends the first part of the proof.

To prove *2*, assume that g is a C^{k+1} satisfying the conditions in *2*. Then f is k-times mu-differentiable. Besides this, $D^k g_{(\cdot)}$ is a C^1 function and

$$\forall a \in {}^{\sigma}U \, \exists \eta_k \approx 0 \, \forall x \approx a \quad |D^k f_x - D^k g_x| \leq \eta_k.$$

By Theorem 4.8, $D^k f_{(\cdot)}$ is mu-differentiable and so f is $(k+1)$-times mu-differentiable. ■

The next theorem establishes a relation between mu-differentiability and a condition similar to SU-differentiability (see Definition 1.21).

Theorem 4.12 *[AN06] For every mu-differentiable function $f : {}^*U \to {}^*F$ we have*

$$\forall x \in ns({}^*U) \, \exists \delta_x \approx 0 \, \exists Df_x \in {}^*L(E,F) \, \forall y \in {}^*U \, \exists \eta \approx 0 \tag{4.1}$$

$$|Df_x| \textit{ is finite } \wedge [\delta_x < |x-y| \approx 0 \Rightarrow f(x) - f(y) = Df_x(x-y) + |x-y|\eta] \, .$$

Proof. For any $x \in ns({}^*U)$, define $a := st(x)$ and $\delta_x := \delta_a$. The proof follows easily. ■

The reverse of Theorem 4.12 is false, as shown in the following example.

Let f be the real valued function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since f is not continuously differentiable, it can not be mu-differentiable. But it satisfies condition (4.1). Indeed, if $x \approx 0$ (the other cases are obvious), for $\delta_x := |x|$ and $y \in {}^*\mathbb{R}$ with $0 \approx |x-y| > \delta_x$, we get

$$\frac{f(x) - f(y)}{x - y} = \frac{x^2}{x-y} \left(\sin \frac{1}{x} - \sin \frac{1}{y} \right) + \frac{x^2 - y^2}{x - y} \sin \frac{1}{y} \approx 0$$

since

$$\left|\frac{x^2}{x-y}\right| \leq \frac{x^2}{|x|} \approx 0 \quad \& \quad \frac{x^2-y^2}{x-y} = x+y \approx 0.$$

As a consequence of the continuity of the derivative, we have (compare with Theorem 1.22)

Theorem 4.13 *[AN06] Let $f : {}^*U \to {}^*F$ be an internal function. Then conditions 1 and 2 are equivalent:*

1. *f is mu-differentiable.*
2. *(a)*

$$\forall a \in {}^\sigma U \, \exists \delta_a \approx 0 \, \exists Df_a \in {}^*L(E,F) \, \forall x, y \in \mu(a)$$

$$|Df_a| \textit{ is finite } \wedge [|x-y| > \delta_a \Rightarrow f(x) - f(y) = Df_a(x-y) + |x-y|\eta]$$

for some $\eta \approx 0$;

(b) $f(ns({}^*U)) \subseteq ns({}^*F)$.

Proof. Let us fix $a \in {}^\sigma U$ and $0 < \delta_a \approx 0$ satisfying

$$\forall x, y \in \mu(a) \quad |x-y| > \delta_a \Rightarrow \frac{f(x)-f(y)}{|x-y|} \approx Df_x\left(\frac{x-y}{|x-y|}\right).$$

By Theorem 4.7 it follows that

$$Df_x\left(\frac{x-y}{|x-y|}\right) \approx Df_a\left(\frac{x-y}{|x-y|}\right)$$

which proves that *1⇒2*.

To prove the converse, let $a \in {}^\sigma U$ and δ_a as in *2(a)* Then, given $x \in \mu(a)$, define $Df_x := Df_a$. The proof follows. ■

Theorem 4.14 *[AN06] If $f : {}^*U \to {}^*F$ is a mu-differentiable function, then for all standard $a \in {}^\sigma U$, there exists a positive $\delta \approx 0$ such that, for all $d \in {}^*E$ with $|d| = 1$, there exists $k \in fin({}^*F)$ for which*

$$\forall x \in {}^*U \quad x \approx a \Rightarrow \frac{f(x+\delta d) - f(x)}{\delta} \approx k$$

holds.

Proof. Fix $a \in {}^{\sigma}U$ and define $\delta := 2\delta_a$. Fix an unit vector d and let $k := Df_a(d)$. Then for $x \approx a$

$$\frac{f(x+\delta d) - f(x)}{\delta} \approx Df_x(d) \approx Df_a(d) = k.$$

■

4.2 The Chain Rule

Theorem 4.15 ***Chain Rule*** *[AN06] Let g and f be two m-differentiable functions at a and $g(a)$, respectively, where a and $g(a)$ are two standard vectors. In addition, if Dg_a is invertible and $|(Dg_a)^{-1}|$ is finite, then $f \circ g$ is m-differentiable at a and $D(f \circ g)_a = Df_{g(a)} \circ Dg_a$.*

Proof. Define $\delta = \max\{\delta_a, 2\delta_{g(a)}|(Dg_a)^{-1}|\}$ and choose x with $\delta < |x-a| \approx 0$.

Since $0 \approx |x-a| > \delta_a$ then $g(x) \approx g(a)$. On the other hand, for some $\eta_1 \approx 0$,

$$|g(x) - g(a)| = |Dg_a(x-a) + |x-a|\eta_1|$$

$$= |x-a| \left| Dg_a\left(\frac{x-a}{|x-a|}\right) + \eta_1 \right| > 2\delta_{g(a)}|(Dg_a)^{-1}| \left| Dg_a\left(\frac{x-a}{|x-a|}\right) + \eta_1 \right| \geq$$

$$2\delta_{g(a)} \left| (Dg_a)^{-1}\left(Dg_a\left(\frac{x-a}{|x-a|}\right) + \eta_1\right) \right| = 2\delta_{g(a)} \left| \frac{x-a}{|x-a|} + (Dg_a)^{-1}(\eta_1) \right| > \delta_{g(a)}.$$

So we conclude that $\delta_{g(a)} < |g(x) - g(a)| \approx 0$. Hence there exists $\eta_2 \approx 0$ such that

$$f(g(x)) - f(g(a)) = Df_{g(a)}(g(x) - g(a)) + |g(x) - g(a)|\eta_2$$

$$= Df_{g(a)}(Dg_a(x-a) + |x-a|\eta_1) + |Dg_a(x-a) + |x-a|\eta_1|\eta_2$$

$$= Df_{g(a)}Dg_a(x-a) + |x-a| \left(Df_{g(a)}(\eta_1) + \left| Dg_a\left(\frac{x-a}{|x-a|}\right) + \eta_1 \right| \eta_2 \right)$$

with

$$Df_{g(a)}(\eta_1) + \left| Dg_a\left(\frac{x-a}{|x-a|}\right) + \eta_1 \right| \eta_2 \approx 0.$$

■

Remark 4.16 Suppose that g and f are two m-differentiable functions at a and $g(a)$, respectively. This is not sufficient to guarantee that $f \circ g$ is also m-differentiable at a, as it will be shown in the following example.

Let ϵ be a positive infinitesimal,

$$\begin{array}{rccc} g: & {}^*\mathbb{R} & \rightarrow & {}^*\mathbb{R} \\ & x & \mapsto & \epsilon x \end{array}$$

and

$$\begin{array}{rccc} f: & {}^*\mathbb{R} & \rightarrow & {}^*\mathbb{R} \\ & x & \mapsto & \begin{cases} 1 & \text{if} \quad 0 < x < \epsilon \\ 0 & \text{if} \quad x \leq 0 \vee x \geq \epsilon \end{cases} \end{array}.$$

It is easy to verify that g is m-differentiable at $x = 0$ and f is m-differentiable at $g(0) = 0$. But

$$\begin{array}{rccc} f \circ g: & {}^*\mathbb{R} & \rightarrow & {}^*\mathbb{R} \\ & x & \mapsto & \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x \leq 0 \vee x \geq 1 \end{cases} \end{array}$$

is not m-differentiable at $x = 0$.

4.3 Taylor's Theorem

We can now formulate Taylor's Theorem for a mu-differentiable function defined on finite dimensional spaces. We will prove two different versions of this theorem; the first Taylor's expansion is made with internal functions and the second with standard functions.

Theorem 4.17 ***Taylor's Theorem*** *[AN06] Let E and F be two standard finite dimensional spaces, U a standard open set and $f : {}^*U \rightarrow {}^*F$ an internal function k-times mu-differentiable, for some $k \in {}^\sigma\mathbb{N}$. Then,*

1. *for every $x \in ns({}^*U)$, there exists $\epsilon \approx 0$ such that, whenever $y \in {}^*U$ with $\epsilon < |y - x| \approx 0$, there exists $\eta \approx 0$ satisfying*

$$f(y) = f(x) + Df_x(y - x) + \frac{1}{2!} D^2 f_x(y - x)^{(2)} + ... + \frac{1}{k!} D^k f_x(y - x)^{(k)} + |y - x|^k \eta.$$

2. *for every $x \in ns({}^*U)$, there exists $\epsilon \approx 0$ such that, whenever $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$, there exists $\eta \approx 0$ satisfying*

$$f(y) = st(f)(x) + Dst(f)_x(y-x) + \frac{1}{2!}D^2 st(f)_x(y-x)^{(2)} + ...$$
$$+\frac{1}{k!}D^k st(f)_x(y-x)^{(k)} + |y-x|^k\eta.$$

Proof.

1. Let us begin by fixing $x \in ns({}^*U)$ and let $a := st(x) \in {}^\sigma U$. By Theorem 4.11, we know that $st(f)$ is of class C^k,

$$\exists \eta_0 \approx 0\, \forall y \approx a \quad |f(y) - st(f)(y)| \leq \eta_0$$

and for each $j = 1, 2, \ldots, k-1$,

$$\exists \eta_j \approx 0\, \forall y \approx a \quad \sup_{d_i \in {}^*E, |d_i|=1} |D^j f_y(d_1, ..., d_j) - D^j st(f)_y(d_1, ..., d_j)| \leq \eta_j.$$

Define $\epsilon = \max\{\eta_0^{\frac{1}{k+1}}, \eta_1^{\frac{1}{k}}, ..., \eta_{k-1}^{\frac{1}{2}}\}$ and take $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$.

Define the finite sequence $(\epsilon_i)_{i=-1,...,k-1}$ by

- $f(y) = st(f)(y) + \epsilon_{-1}$,
- $f(x) = st(f)(x) + \epsilon_0$,
- $Df_x(y-x) = Dst(f)_x(y-x) + |y-x|\epsilon_1$,
- $D^2 f_x(y-x)^{(2)} = D^2 st(f)_x(y-x)^{(2)} + |y-x|^2\epsilon_2$,
- ...
- $D^{k-1} f_x(y-x)^{(k-1)} = D^{k-1} st(f)_x(y-x)^{(k-1)} + |y-x|^{k-1}\epsilon_{k-1}$.

Furthermore, since the maps $x \mapsto D^k f_x$ and $x \mapsto D^k st(f)_x$ are both S-continuous, we also have

$$D^k f_x\left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k f_a\left(\frac{y-x}{|y-x|}\right)^{(k)} \approx$$
$$D^k st(f)_a\left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k st(f)_x\left(\frac{y-x}{|y-x|}\right)^{(k)},$$

so there exists $\epsilon_k \approx 0$ with

$$D^k f_x(y-x)^{(k)} = D^k st(f)_x(y-x)^{(k)} + |y-x|^k \epsilon_k.$$

Using the fact that $st(f)$ is a C^k function, one has

$$st(f)(y) = st(f)(x) + Dst(f)_x(y-x) + \frac{1}{2!}D^2 st(f)_x(y-x)^{(2)} + ...$$
$$+\frac{1}{k!}D^k st(f)_x(y-x)^{(k)} + |y-x|^k \eta,$$

that is

$$f(y) = f(x) + Df_x(y-x) + \frac{1}{2!}D^2 f_x(y-x)^{(2)} + ... + \frac{1}{k!}D^k f_x(y-x)^{(k)} + |y-x|^k \eta$$
$$+\epsilon_{-1} - \epsilon_0 - |y-x|\epsilon_1 - |y-x|^2\epsilon_2 - ... - |y-x|^{k-1}\epsilon_{k-1} - |y-x|^k \epsilon_k.$$

If

$$\epsilon_{-1} - \epsilon_0 - |y-x|\epsilon_1 - |y-x|^2\epsilon_2 - ... - |y-x|^{k-1}\epsilon_{k-1} = |y-x|^k \eta_1,$$

then η_1 is infinitesimal since

$$|\eta_1| \leq \frac{|\epsilon_{-1}|}{|y-x|^k} + \frac{|\epsilon_0|}{|y-x|^k} + \frac{|\epsilon_1|}{|y-x|^{k-1}} + \frac{|\epsilon_2|}{|y-x|^{k-2}} + \ldots + \frac{|\epsilon_{k-1}|}{|y-x|}$$
$$\leq \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_1}{\eta_1^{\frac{k-1}{k}}} + \frac{\eta_2}{\eta_2^{\frac{k-2}{k-1}}} + ... + \frac{\eta_{k-1}}{\eta_{k-1}^{\frac{1}{2}}} \approx 0.$$

2. Analogously, if we take $\epsilon := \eta_0^{\frac{1}{k+1}}$, the result follows.

■

4.4 The Mean Value Theorem

We give now a Mean Value Theorem for mu-differentiable functions.

Theorem 4.18 ***Mean Value Theorem*** *[AN06] Let U be a standard open convex subset of E and $f : {}^*U \to {}^*\mathbb{R}$ an internal mu-differentiable function. Then, for all $x, y \in ns({}^*U)$ with $|x-y| > \delta_a$, where $a := st(x)$*

$$\exists c \in [x, y] \quad f(x) - f(y) = Df_c(x-y) + |x-y|\eta$$

for some $\eta \approx 0$.

Proof. Define a hyper-finite sequence $\{x_n \mid n \in \{1, \ldots, N+1\}\}$, for some $N \in {}^*\mathbb{N}$, in the following way.

First, let $x_1 := x$. Since f is mu-differentiable, by Theorem 4.12 there exist a positive infinitesimal δ_1 and a finite linear operator Df_{x_1} such that, for all $z \in {}^*U$:

$$\delta_1 < |z - x_1| \approx 0 \Rightarrow f(z) - f(x_1) = Df_{x_1}(z - x_1) + |z - x_1|\eta_1,$$

for some $\eta_1 \approx 0$.

Now, let $x_2 := x_1 + 2\delta_1 \dfrac{y-x}{|y-x|}$. Since $\delta_1 < |x_2 - x_1| \approx 0$, then

$$f(x_2) - f(x_1) = Df_{x_1}(x_2 - x_1) + |x_2 - x_1|\eta_1.$$

Similarly, there exists δ_2 (suppose $\delta_2 > \delta_1$) with, for all $z \in {}^*U$:

$$\delta_2 < |z - x_2| \approx 0 \Rightarrow f(z) - f(x_2) = Df_{x_2}(z - x_2) + |z - x_2|\eta_2$$

and define $x_3 := x_2 + 2\delta_2 \dfrac{y-x}{|y-x|}$.

Repeating the process, we obtain a sequence $\{x_n | 1 \leq n \leq N+1\}$ which satisfies the conditions

- $x_1 = x$;

- $x_{n+1} = x_n + 2\delta_n \dfrac{y-x}{|y-x|}$, $\delta_n \approx 0$ and $\delta_n > \delta_1$, $n = 1, \ldots, N$;
- $f(x_{n+1}) - f(x_n) = Df_{x_n}(x_{n+1} - x_n) + |x_{n+1} - x_n|\eta_n$, for some $\eta_n \approx 0$, $n = 1, \ldots, N$;
- $x_{N+1} = y$ (if not, choose $0 \approx \delta > \delta_N$ with $x_N + 2\delta \dfrac{y-x}{|y-x|} = y$).

Then

$$f(x) - f(y) = \sum_{n=1}^{N} (f(x_n) - f(x_{n+1})) =$$

$$= \sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1}) + \sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n.$$

If

$$\sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n = |y-x|\eta,$$

for some η, then $\eta \approx 0$. Indeed, by the convexity property of the norm

$$|\eta| \leq \frac{\sum_{n=1}^{N} |x_n - x_{n+1}||\eta_n|}{|y-x|} = \frac{\sum_{n=1}^{N} |x_n - x_{n+1}||\eta_n|}{\sum_{n=1}^{N} |x_n - x_{n+1}|} \leq \max_{n \in \{1,\ldots,N\}} \{|\eta_n|\} \approx 0.$$

We will prove now that there exists $c \in [x, y]$ such that

$$Df_c\left(\frac{x-y}{|x-y|}\right) \approx \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x-y|}.$$

Letting $d := \dfrac{x-y}{|x-y|}$, it is true that

$$\frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x-y|} = \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{\sum_{n=1}^{N} |x_n - x_{n+1}|} = \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n}.$$

Choosing $m, M \in \{x_1, ..., x_N\}$ with

$$Df_m(d) = \min_{1 \leq n \leq N} Df_{x_n}(d) \quad \& \quad Df_M(d) = \max_{1 \leq n \leq N} Df_{x_n}(d),$$

we get

$$Df_m(d) \leq \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n} \leq Df_M(d).$$

So, there exists $c \in [m, M] \subseteq [x, y]$ with

$$Df_c(d) \approx \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n}.$$

∎

We can formulate Theorem 4.18 for functions taking values in a normed space:

Theorem 4.19 *[AN06] Let U be a standard open convex subset of E and $f : {}^*U \to {}^*F$ an internal mu-differentiable function. Then, for all $x, y \in ns({}^*U)$ with $|x-y| > \delta_a$, where $a := st(x)$*

$$\exists c \in [x, y] \quad |f(x) - f(y)| \leq |Df_c(x-y)| + |x-y|\eta$$

for some $\eta \approx 0$.

Proof. Following the proof of Theorem 4.18, it is true that:

For some $\eta_1, ..., \eta_N \approx 0$,

$$|f(x) - f(y)| = \left| \sum_{n=1}^{N} (f(x_n) - f(x_{n+1})) \right|$$

$$\leq \sum_{n=1}^{N} |f(x_n) - f(x_{n+1})| = \sum_{n=1}^{N} |Df_{x_n}(x_n - x_{n+1}) + |x_n - x_{n+1}|\eta_n|$$

$$\leq \sum_{n=1}^{N} 2\delta_n \, |Df_{x_n}(d)| + \sum_{n=1}^{N} |x_n - x_{n+1}||\eta_n|.$$

Again, choose $m, M \in \{x_1, ..., x_N\}$ with

$$|Df_m(d)| = \min_{1 \leq n \leq N} |Df_{x_n}(d)| \quad \& \quad |Df_M(d)| = \max_{1 \leq n \leq N} |Df_{x_n}(d)|.$$

Since

$$|Df_m(d)| \leq \frac{\sum_{n=1}^{N} 2\delta_n |Df_{x_n}(d)|}{\sum_{n=1}^{N} 2\delta_n} \leq |Df_M(d)|$$

there exists $c \in [x, y]$ with

$$|Df_c(d)| \approx \frac{\sum_{n=1}^{N} 2\delta_n |Df_{x_n}(d)|}{\sum_{n=1}^{N} 2\delta_n}.$$

∎

4.5 The Inverse Mapping Theorem

A full Inverse Mapping Theorem is not expected. In fact, take for example the C^1 function $g(x) = x$. By Theorem 4.8, any internal function infinitely close to g is mu-differentiable. So the $1-1$ condition may easily fail. Nevertheless, we have some form of injectivity as the next theorem states.

Theorem 4.20 ***Inverse Mapping Theorem*** *[AN06] Let $f : {}^*U \to {}^*F$ be an internal mu-differentiable function. Assume that, for a certain $a \in {}^\sigma U$, Df_a is invertible and $|(Df_a)^{-1}|$ is finite. Then there exists a standard neighborhood ${}^*V$ of a such that f is* 1*-to-*1 *on the standard elements of ${}^*V$, i.e.,*

$$\forall x, y \in {}^\sigma V \;\; x \neq y \Rightarrow f(x) \neq f(y).$$

Proof. Let

$$A := \left\{ \epsilon \in {}^*\mathbb{R}^+ \,|\, \forall x, y \in B_\epsilon(a) \quad |x - y| > \delta_a \Rightarrow f(x) \neq f(y) \right\}.$$

Then A contains all positive infinitesimal numbers since, for $0 < \epsilon \approx 0$ and $x, y \in B_\epsilon(a)$ with $|x - y| > \delta_a$, by Theorem 4.13,

$$\frac{f(x) - f(y)}{|x - y|} \approx Df_a \left(\frac{x - y}{|x - y|} \right).$$

But

$$1 = \left| (Df_a)^{-1} Df_a \left(\frac{x - y}{|x - y|} \right) \right| \leq \left| (Df_a)^{-1} \right| \left| Df_a \left(\frac{x - y}{|x - y|} \right) \right|.$$

Consequently,

$$\left| Df_a \left(\frac{x - y}{|x - y|} \right) \right| \geq \frac{1}{|(Df_a)^{-1}|} \not\approx 0.$$

Therefore $f(x) \neq f(y)$. Using the Spillover Principle we can guarantee the existence of $\epsilon \in {}^\sigma\mathbb{R}$ with $\epsilon \in A$. Define $V := B_\epsilon(a)$ and take two standard elements of ${}^*V$ with $x \neq y$. Since the distance between two distinct standard vectors is always greater than any infinitesimal number, one obtains $f(x) \neq f(y)$. ■

Remark 4.21 With the previous conditions we can not conclude that f is 1-to-1 on ${}^*V$. In fact, consider

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ \epsilon & \text{if } x = 0 \end{cases}$$

where ϵ is any non-zero infinitesimal number. This function is mu-differentiable (it is infinitely close to $g(x) = x$) but it is never injective in any standard neighborhood of zero.

Chapter 5

Infinitesimal Differential Geometry

Differential Geometry seems to be a natural field for application of nonstandard analysis. In *Disquisitiones generales circa superficies curvas* (1827), Gauss stated

A curved surface is said to possess continuous curvature at one of its points A, if the directions of all straight lines drawn from A to points of the surface at an infinite small distance from A are deflected infinitely little from one and the same plane passing through A (taken from [Str77]).

Interesting enough very few work has been done in this direction; the interested reader can look into [Cos01], [Goz95], [Dra98], [HJ00], [HJ01], [KR98] and [Str77].

5.1 Cusps

Let $\alpha : I \to \mathbb{R}^n$ be a curve (possibly with side derivatives α'_+ and α'_-). In the literature we may find two distinct definitions of **cusp**:

1. $\alpha(t)$ is called a cusp if $\alpha'_+(t) = -\alpha'_-(t) \neq 0$;

2. $\alpha(t)$ is called a cusp if $\alpha'(t) = 0$ and $\alpha''(t) \neq 0$.

We begin this section presenting four nonstandard definitions of cusp. Later we establish some relations between them and with Definitions 1 and 2 above.

Let a be a positive real number, $\alpha :]-a, a[\to \mathbb{R}^n (n \geq 2)$ be a continuous curve such that

$$\alpha(0) = 0 \text{ and } \alpha \text{ is } 1-1.$$

Define

$$\rho(t) := \frac{|\alpha(t) - \alpha(-t)|}{|\alpha(t)|} \quad (t \neq 0).$$

Let $\theta(t)$ the angle between $\alpha(t)$ and $\alpha(-t)$

$$\theta(t) := \angle\, \alpha(t)\, 0\, \alpha(-t),$$

so that

$$\cos(\theta(t)) = \frac{\alpha(t) \cdot \alpha(-t)}{|\alpha(t)| \cdot |\alpha(-t)|} \quad (t \neq 0).$$

Let $R(t)$ be the ratio

$$R(t) := \frac{|\alpha(-t)|}{|\alpha(t)|} \quad (t \neq 0).$$

Definition 5.1

1. *$\alpha(0)$ is a* **Vector – Cusp** *(or short* **V – Cusp***) if the following condition is verified.*

$$\exists u \in \mathbb{R}^n \left[|u| = 1 \,\&\, \forall\, 0 \neq \epsilon \approx 0 \quad \frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx u \right]. \tag{5.1}$$

2. *$\alpha(0)$ is a* **symmetric V – Cusp** *if the following condition is verified*

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|} \text{ for all positive } \epsilon \approx 0. \tag{5.2}$$

3. *$\alpha(0)$ is a* **Triangle – Cusp** *(or short* **T – Cusp***) if the following condition is verified*

$$\rho(\epsilon) \approx 0, \text{ for all non-zero } \epsilon \approx 0. \tag{5.3}$$

4. *$\alpha(0)$ is a* **$\mathbf{T}^+$ – Cusp** *if the following condition is verified*

$$\rho(\epsilon) \approx 0, \text{ for all positive } \epsilon \approx 0. \tag{5.4}$$

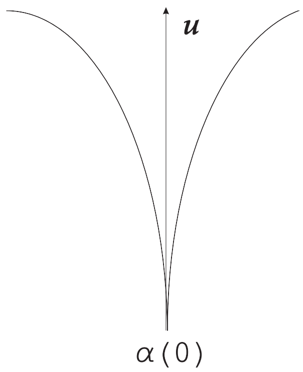

Figure 5.1: V-Cusp

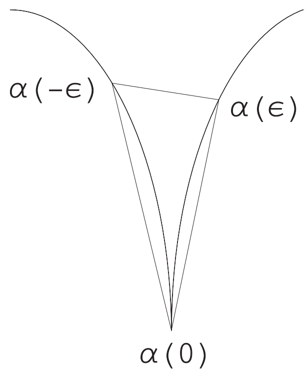

Figure 5.2: T-Cusp

For example, let $\alpha(t) = (t^2, t^3), t \in \mathbb{R}$. Then

1. $\alpha(0)$ is a V-cusp (Figure 5.1) since, for every non-zero ϵ,

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} = \frac{(1,\epsilon)}{|(1,\epsilon)|} \approx (1,0).$$

2. $\alpha(0)$ is a T-cusp (Figure 5.2) since, for $0 \neq \epsilon \approx 0$,

$$\frac{|\alpha(\epsilon) - \alpha(-\epsilon)|}{|\alpha(\epsilon)|} = \frac{|(0,2\epsilon)|}{|(1,\epsilon)|} = \frac{2|\epsilon|}{\sqrt{1+\epsilon^2}} \approx 0.$$

Remark: Given a 1-1 continuous curve α, defined in a neighborhood of some real number t_0, $\alpha :]t_0 - a, t_0 + a[\to \mathbb{R}^n$, $\alpha(t_0)$ is, by definition, a **Cusp** `of any of the previously defined types` (vide Definition 5.1) if $\beta(0)$ is, where $\beta(s) = \alpha(s + t_0) - \alpha(t_0)$ $(|s| < a)$.

All definitions and all results proved henceforth have then an equivalent version with *adequate substitutions* of t_0 for 0, and therefore of $t_0 \pm \epsilon$ for $\pm\epsilon$ respectively.

Assume that there exist

$$\alpha'_+(0) := st\left(\frac{\alpha(\epsilon)}{\epsilon}\right) \ (0 < \epsilon \approx 0) \text{ and } \alpha'_-(0) = st\left(\frac{\alpha(\epsilon)}{\epsilon}\right) \ (0 > \epsilon \approx 0).$$

Theorem 5.2 *[AN05a]*

1. (5.1) $\Rightarrow$ (5.2);

2. (5.1) $\not\Leftarrow$ (5.2), (5.1) $\not\Leftarrow$ (5.3) *and* (5.1) $\not\Rightarrow$ (5.3);

3. *If* $\alpha'_+(0) \neq 0 \neq \alpha'_-(0)$,

$$(5.1) \Leftrightarrow (5.2) \Leftrightarrow \frac{\alpha'_+(0)}{|\alpha'_+(0)|} = -\frac{\alpha'_-(0)}{|\alpha'_-(0)|}. \tag{5.5}$$

Proof. Condition 1 is obvious.

On what regards condition 2, we proceed to describe a continuous 1-1 curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ which verifies (5.2) and (5.3) but not (5.1); simply define

$$\alpha(t) = \begin{cases} (t, t\sin\left(\frac{1}{t}\right), t^2) & t > 0 \\ (0,0,0) & t = 0 \\ (-t, t\sin\left(\frac{1}{t}\right), -t^2) & t < 0 \end{cases}$$

To see that (5.1) $\not\Rightarrow$ (5.3), let β be given by

$$\beta(t) = \begin{cases} (t^6, t^4) & t \geq 0 \\ (t^3, t^2) & t < 0 \end{cases}$$

This curve verifies (5.1) but not (5.3).

For condition 3, consider the following. When $0 \neq \epsilon \approx 0$, , whatever ϵ, or for that matter whatever its sign might be,

$$\begin{aligned} \frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} &= \frac{\epsilon}{|\alpha(\epsilon)|} \cdot \frac{\alpha(\epsilon)}{\epsilon} \\ &= \operatorname{sgn}(\epsilon) \frac{\alpha(\epsilon)/\epsilon}{|\alpha(\epsilon)/\epsilon|} \\ &\approx \begin{cases} \frac{\alpha'_+(0)}{|\alpha'_+(0)|} & \epsilon > 0 \\ -\frac{\alpha'_-(0)}{|\alpha'_-(0)|} & \epsilon < 0 \end{cases} \end{aligned}$$

(where $\operatorname{sgn}(\epsilon) = \epsilon/|\epsilon|$).

Therefore the two standard $\dfrac{\alpha'_+(0)}{|\alpha'_+(0)|}, -\dfrac{\alpha'_-(0)}{|\alpha'_-(0)|}$ are equal if and only if they are infinitely close if and only if for all positive $\epsilon \approx 0$, $\dfrac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \dfrac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|}$ if and only if all the $\dfrac{\alpha(\epsilon)}{|\alpha(\epsilon)|}$ have the same standard part. ∎

Theorem 5.3 *[AN05a]*

1. *Conditions* (5.3), (5.4) *and the following are equivalent.*

$$\theta(\epsilon) \approx 0 \textit{ and } R(\epsilon) \approx 1, \textit{ whenever } 0 \neq \epsilon \approx 0; \tag{5.6}$$

2. (5.3) $\Rightarrow$ (5.2) *but* (5.2) $\not\Rightarrow$ (5.3).

Proof.

1. According to the law of cosines, applied to the triangle with vertices 0, $\alpha(\epsilon)$, $\alpha(-\epsilon)$, whatever the sign of ϵ might be,

$$|\alpha(\epsilon) - \alpha(-\epsilon)|^2 = |\alpha(\epsilon)|^2 + |\alpha(-\epsilon)|^2 - 2|\alpha(\epsilon)| \cdot |\alpha(-\epsilon)| \cdot \cos(\theta(\epsilon))$$

so that

$$\begin{aligned} \rho(\epsilon)^2 &= 1 + R(\epsilon)^2 - 2R(\epsilon)\cos(\theta(\epsilon)) \\ &= (1 - R(\epsilon))^2 \ + \ 2R(\epsilon)(1 - \cos(\theta(\epsilon))). \end{aligned}$$

As $(1 - R(\epsilon))^2$, $2R(\epsilon)$, $(1 - \cos(\theta(\epsilon)))$ are all nonnegative and $\rho(-\epsilon) = \rho(\epsilon)\dfrac{1}{R(\epsilon)}$,

$$(5.3) \Leftrightarrow (5.4) \Leftrightarrow (5.6)$$

follows.

2. Observe that

$$\begin{aligned} \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} - \frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|}\right| &= \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} - \frac{|\alpha(\epsilon)|}{|\alpha(-\epsilon)|} \cdot \frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right| \\ &= \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} - \frac{1}{R(\epsilon)} \cdot \frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right|. \end{aligned}$$

If $\rho(\epsilon) \approx 0$, then $\dfrac{1}{R(\epsilon)} \approx 1$ and $\dfrac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}$ is also finite (actually its norm is precisely $R(\epsilon)$) so that

$$\left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} - \frac{1}{R(\epsilon)} \cdot \frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right| \approx \left|\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} - \frac{\alpha(-\epsilon)}{|\alpha(\epsilon)|}\right| = \rho(\epsilon) \approx 0.$$

and (5.3) $\Rightarrow$ (5.2) is proven. Finally, the curve β defined in the proof of Theorem 5.2 verifies (5.2) but not (5.3). ∎

Theorem 5.4 *[AN05a] If $\alpha'_+(0) \neq 0$ (or $\alpha'_-(0) \neq 0$) then condition* (5.3) *is equivalent to the following.*

$$\alpha'_+(0) = -\alpha'_-(0). \tag{5.7}$$

Proof. For $0 < \epsilon \approx 0$, there exist $\eta, \iota \approx 0$ such that

$$\begin{aligned} \alpha(\epsilon) - \alpha(-\epsilon) &= \alpha'_+(0)\epsilon + \epsilon\eta \ + \ \alpha'_-(0)\epsilon + \epsilon\iota \\ &= (\alpha'_+(0) + \alpha'_-(0))\epsilon \ + \ (\eta + \iota)\epsilon \end{aligned}$$

and

$$\rho(\epsilon) = \frac{|(\alpha'_+(0) + \alpha'_-(0)) + (\eta + \iota)|}{|\alpha'_+(0) + \eta|}.$$

As $\alpha'_+(0)$ is standard and non-zero,

$$\rho(\epsilon) \approx 0 \text{ if and only if } \alpha'_+(0) + \alpha'_-(0) \approx 0.$$

Since $\alpha'_-(0)$ is also standard,

$$\rho(\epsilon) \approx 0 \text{ if and only if } \alpha'_+(0) + \alpha'_-(0) = 0.$$

as required. ∎

Theorem 5.5 *[AN05a] If α is of class $C^2, \alpha'(0) = 0$ and $\alpha''(0) \neq 0$, then $\alpha(0)$ is a* $\mathbf{T} - \mathbf{cusp}$ *and a* $\mathbf{V} - \mathbf{cusp}$.

Proof. For each positive infinitesimal ϵ, there exist $\eta, \iota \approx 0$ such that

$$\alpha(\epsilon) - \alpha(-\epsilon) = \epsilon^2(\eta - \iota);$$

and therefore

$$\rho(\epsilon) = \frac{|\eta - \iota|}{\left|\frac{\alpha''(0)}{2} + \eta\right|} \approx 0.$$

Furthermore, we have

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha''(0)}{|\alpha''(0)|}$$

for all non-zero infinitesimal ϵ. ∎

More generally, the following is a simple application of Taylor's Theorem.

Theorem 5.6 *[AN05a] Let α be a curve of class C^{2k+1}.*

1. *If $\alpha(0)$ is a* **T − Cusp** *(or a* **V − Cusp***) and, for all $i \in \{1, \cdots, k\}$, $\alpha^{(2i)}(0) = 0$, then for all $i \in \{1, \cdots, 2k+1\}$ $\alpha^{(i)}(0) = 0$.*
2. *If for all $i \in \{1, \cdots, k\}$, $\alpha^{(2i-1)}(0) = 0$ and $\alpha^{(2k)}(0) \neq 0$, then*
   (a) $\alpha(0)$ is a **T − Cusp***,*
   (b) $\alpha(0)$ is a **V − Cusp** *and, for all infinitesimal ϵ,*

   $$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha^{(2j)}(0)}{|\alpha^{(2j)}(0)|}$$

   where

   $$j := \min\{i \in \{1, \ldots, k\} \mid \alpha^{(2i)}(0) \neq 0\}.$$

Proof.

1. We know that $\alpha'(0) = 0$; assume that α is C^{2k+1} and

$$\alpha^{(j)}(0) = 0 \quad (0 \leq j \leq 2k).$$

Then, for some infinitesimals η_1, η_2,

$$0 \approx \rho(\epsilon) = \frac{|\alpha(\epsilon) - \alpha(-\epsilon)|}{|\alpha(\epsilon)|}$$

$$= \frac{\left|\frac{2}{(2k+1)!}\alpha^{(2k+1)}(0) + (\eta_1 + \eta_2)\right|}{\left|\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0) + \eta_1\right|},$$

for all non-zero infinitesimal ϵ.

This can happen only if the standard $\alpha^{(2k+1)}(0)$ is 0.

Assume now that $\alpha(0)$ is a $\mathbf{V}-\mathbf{cusp}$. Similarly, we have

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|}=\operatorname{sgn}(\epsilon)\frac{\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0)+\eta}{\left|\frac{1}{(2k+1)!}\alpha^{(2k+1)}(0)+\eta\right|}\approx u$$

for some $u\in\mathbb{R}^n$ and all ϵ only if $\alpha^{(2k+1)}(0)=0$.

2. *(a)* is a straightforward adaptation of the proof above or of the proof of Theorem 5.5;

(b) is a very obvious application of Taylor's formula for there exists $\iota\approx 0$ such that

$$\alpha(\epsilon)=\frac{\epsilon^{2j}}{(2j)!}\left(\alpha^{(2j)}(0)+\iota\right).$$

■

Actually the conditions in Definition 5.1 make sense even if one of the lateral derivatives does not exist:

Theorem 5.7 *[AN05a] If α is the graph of a function $f:]-a,a[\to\mathbb{R}$, then the following are equivalent.*

1. $(0,0)=(0,f(0))$ *is a* $\mathbf{T}-\mathbf{Cusp}$.
2. *The right and left derivatives* $f'_+(0),\ f'_-(0)$ *are both infinite with opposite signs.*

Proof. In this case, for each non-zero infinitesimal ϵ,

$$\rho(\epsilon)=\sqrt{\frac{4+\left(\frac{f(\epsilon)}{\epsilon}-\frac{f(-\epsilon)}{\epsilon}\right)^2}{1+\left(\frac{f(\epsilon)}{\epsilon}\right)^2}}. \tag{5.8}$$

Define

$$\Omega=\frac{f(\epsilon)}{\epsilon}\quad\&\quad\Phi=\frac{f(-\epsilon)}{-\epsilon}$$

The numerator of (5.8) is not infinitesimal, therefore $\rho(\epsilon)\approx 0$ if and only if the two following conditions are verified.

1. Ω is infinite.

2. $\rho(\epsilon)^2 = \frac{\frac{4}{\Omega^2}+\left(1+\frac{\Phi}{\Omega}\right)^2}{\frac{1}{\Omega^2}+1} \approx 0.$

which , in turn, are equivalent to

1. Ω is infinite.

2. $\frac{\Phi}{\Omega} \approx -1.$

which is even stronger than required. ■

5.2 Regular Cusps

Suppose further that α is obtained from two regular C^1 curves with at least C^1 contact at 0, that is, there exist two regular C^1 curves $\beta, \delta :]-a, a[\to \mathbb{R}^n$ such that

$$\begin{array}{rcl} \alpha(t) & = & \beta(t) \quad (-a < t \leq 0) \\ \alpha(t) & = & \delta(t) \quad (0 \leq t < a) \\ \beta'(0) & \text{and} & \delta'(0) \text{ are collinear.} \end{array}$$

If this is the case, α is **rectifiable**, in the sense that there exists an arc-length function

$$s(t) := \int_0^t |\alpha'(\tau)|\, d\tau \quad (t \in]-a, a[). \tag{5.9}$$

If we also assume that α is parameterized by this arc-length, $\alpha'(0)$ might not exist, but

$$\begin{array}{rcl} |\alpha'_+(0)| & = & 1 \\ |\alpha'_-(0)| & = & 1 \\ \alpha'_+(0) & = & \pm\alpha'_-(0). \end{array}$$

Therefore

Theorem 5.8 *[AN05a] If α is parameterized by the arc-length defined in equation* (5.9)*, then all conditions in Definition 5.1 are equivalent and equivalent to* (5.6) *and* (5.7)*.*

Proof. We only need to prove that (5.2) $\Rightarrow$ (5.3). Let $\epsilon \approx 0$; then

$$\frac{\alpha(\epsilon)}{|\alpha(\epsilon)|} \approx \frac{\alpha(-\epsilon)}{|\alpha(-\epsilon)|} \Rightarrow \alpha'_+(0) = -\alpha'_-(0) \Rightarrow (5.3).$$

■

5.3 Envelopes of Families of Curves

Consider a family of curves $\{\alpha_\lambda \,|\, \lambda \in I\}$. As usual, an **envelope** of this family is a curve which at each of its points is tangent to a curve of the family. Suppose that the family of curves $\{\alpha_\lambda :]a,b[\to \mathbb{R}^2 \,|\, \lambda \in I\}$ is given by the equation $F(x,y,\lambda) = 0$, where F is a C^1 real valued function. The envelope is the result of eliminating λ between the two equations

$$F(x,y,\lambda) = 0 \text{ and } \frac{\partial F}{\partial \lambda}(x,y,\lambda) = 0.$$

For example, if $F(x,y,\lambda) = (x-\lambda)^2 + y^2 - 1$, then the solution is $y = \pm 1$ (see Figure 5.3). For more details, see [Agu92].

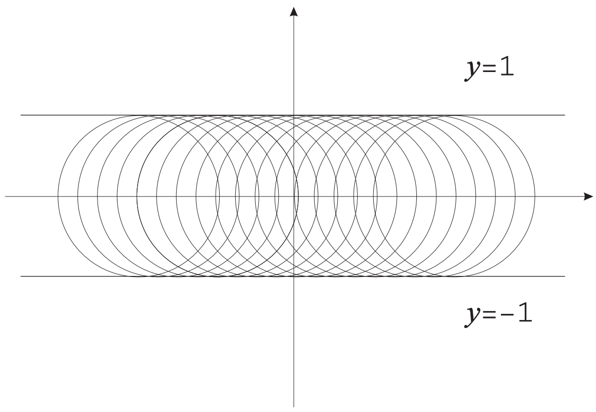

Figure 5.3

Let α_λ and $\alpha_{\lambda'}$ be two curves of this family. If they are near enough, they will meet in two distinct points; these points will be infinitely close to the lines $y = \pm 1$, if α_λ is infinitely close to $\alpha_{\lambda'}$. More precisely:

Let I and J be open intervals of $\mathbb{R}$ and Λ be a family of C^1 curves $\alpha_\lambda : J \to \mathbb{R}^n$,

$$\Lambda := \{\alpha_\lambda | \, \lambda \in I\},$$

and define $F : J \times I \to \mathbb{R}^n$ by

$$F(t, \lambda) := \alpha_\lambda(t).$$

Theorem 5.9 *[AN05a] Suppose that*

1. *F is of class C^1.*
2. *There exists a C^1 function $f : I \to \mathbb{R}$ which satisfies the following: for all $\lambda \in I$ and $\delta \approx 0$, there exists a pair $(t, t') \in {}^*J^2$, such that*

$$\begin{aligned} t &\approx t' \approx f(\lambda) \\ \alpha_\lambda(t) &= \alpha_{\lambda+\delta}(t'). \end{aligned} \tag{5.10}$$

Define

$$\beta(\lambda) := \alpha_\lambda(f(\lambda)) = F(f(\lambda), \lambda) \quad (\lambda \in I).$$

Then $\beta'(\lambda)$ and $\alpha'_\lambda(f(\lambda))$ are collinear and thus β is an envelope of Λ.

Proof. Consider that

$$\beta'(\lambda) = f'(\lambda)\frac{\partial F}{\partial t}(f(\lambda), \lambda) + \frac{\partial F}{\partial \lambda}(f(\lambda), \lambda). \tag{5.11}$$

Next take $\lambda \in I$, a non-zero $\delta \approx 0$, the t, t' given by condition (5.10) and observe that

$$F(t', \lambda + \delta) - F(t, \lambda) = 0$$

so that, there exists $\eta \approx 0$, such that

$$DF_{(t,\lambda)}(t' - t, \delta) + |(t' - t, \delta)|\eta = 0.$$

Therefore

$$DF_{(t,\lambda)}\left(\frac{(t' - t, \delta)}{|(t' - t, \delta)|}\right) = -\eta \;\approx\; 0.$$

As F is C^1, it follows

$$DF_{(f(\lambda),\lambda)}\left(st\left(\frac{(t'-t,\delta)}{|(t'-t,\delta)|}\right)\right)=0,$$

hence

$$\frac{\partial F}{\partial t}(f(\lambda),\lambda) \text{ and } \frac{\partial F}{\partial \lambda}(f(\lambda),\lambda) \text{ are collinear.} \tag{5.12}$$

Finally, note that

$$\alpha'_\lambda(f(\lambda))=\frac{\partial F}{\partial t}(f(\lambda),\lambda). \tag{5.13}$$

Conditions (5.11), (5.12) and (5.13) imply that $\beta'(\lambda)$ and $\alpha'_\lambda(f(\lambda))$ are collinear. ■

Example Let

$$\alpha_\lambda(t):=\left(t,\lambda t+\lambda^2,\frac{\lambda}{2}t+\frac{\lambda^2}{2}\right)\ (\lambda,t\in\mathbb{R}).$$

Then

$$\alpha_\lambda(t)=\alpha_{\lambda+\delta}(t')\Leftrightarrow t=t'=-2\lambda-\delta.$$

If $\delta\approx 0$ then $t=t'\approx -2\lambda$. It follows that $f(\lambda):=-2\lambda$ and the envelope is the curve $\lambda\overset{\beta}{\mapsto}(-2\lambda,-\lambda^2,-\frac{\lambda^2}{2})$ (Figure 5.4).

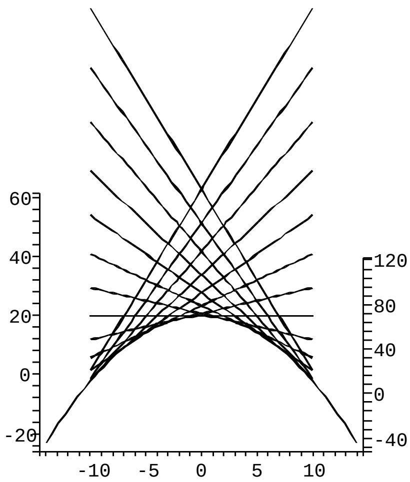

Figure 5.4

5.4 The Coffeecup Caustic

The Coffeecup Caustic is the (planar) envelope of the family of (co-planar) light rays reflected on a concave semi-cylindrical mirror, from a light source located (on the plane of the rays) at an infinite distance from the mirror, so that the produced light rays are parallel. For more detailed information, we refer [BGG81] and [BGG84].

We shall assume the mirror is the upper half-circle

$$(\cos(\lambda), \sin(\lambda)) \quad (0 < \lambda < \pi)$$

and that the incident rays are parallel to the y axis, with light source at $-\infty$ (Figure 5.5).

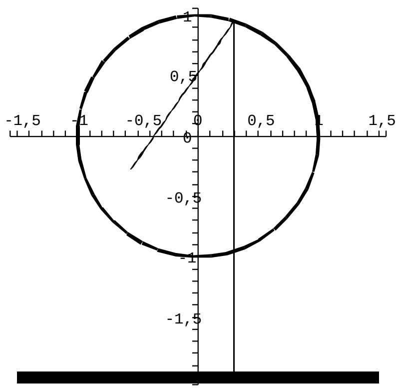

Figure 5.5

The reflected rays are the half-lines

$$\alpha_\lambda(t) = (\cos\lambda, \sin\lambda) + t(-\sin(2\lambda), \cos(2\lambda)), \quad t > 0, \lambda \in]0, \pi[.$$

The equation

$$\alpha_\lambda(t) = \alpha_{\lambda+\delta}(t')$$

has solution

$$\begin{aligned} t &= \frac{\cos(\lambda+\delta) - \cos(\lambda+2\delta)}{\sin(2\delta)} \\ t' &= \frac{\cos(\lambda-\delta) - \cos(\lambda)}{\sin(2\delta)}. \end{aligned}$$

Therefore, for some $\theta, \tau \in {}^*]0,1[$ and $\delta \approx 0$,

$$\begin{aligned} t &= \frac{\sin(\lambda + (2-\theta)\delta)\delta}{\sin(2\delta)} \approx \frac{\sin(\lambda)}{2} \\ t' &= \frac{\sin(\lambda - \tau\delta)\delta}{\sin(2\delta)} \approx \frac{\sin(\lambda)}{2}. \end{aligned}$$

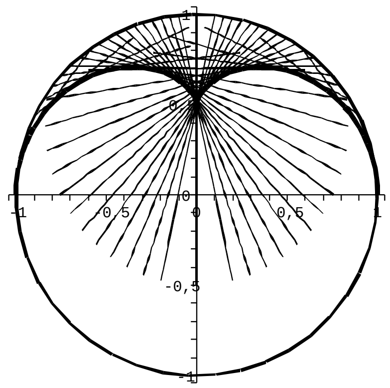

Figure 5.6

Taking

$$f(\lambda) := \frac{\sin(\lambda)}{2},$$

Theorem 5.9 says that

$$\lambda \mapsto \beta(\lambda) := \left(\cos^3(\lambda), \sin(\lambda) + \frac{\sin(\lambda)}{2}\cos(2\lambda)\right)$$

is an envelope of the α_λ (Figure 5.6).

Also note that $\beta(\frac{\pi}{2}) = (0, \frac{1}{2})$ is a $\mathbf{T}-\mathbf{cusp}$ because, for $0 < \epsilon \approx 0$,

$$\begin{aligned} \beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}-\epsilon\right) &= (-2\sin^3(\epsilon), 0) \\ \beta\left(\frac{\pi}{2}+\epsilon\right) - \beta\left(\frac{\pi}{2}\right) &= \left(-\sin^3(\epsilon), \cos(\epsilon) - \frac{\cos(\epsilon)}{2}\cos(2\epsilon) - \frac{1}{2}\right) \\ &= \left(-\sin^3(\epsilon), \sin^2(\epsilon)\left(\cos(\epsilon) - \frac{1}{2(\cos(\epsilon)+1)}\right)\right). \end{aligned}$$

Therefore

$$\left(\frac{|\beta\left(\frac{\pi}{2}+\epsilon\right)-\beta\left(\frac{\pi}{2}-\epsilon\right)|}{|\beta\left(\frac{\pi}{2}+\epsilon\right)-\beta\left(\frac{\pi}{2}\right)|}\right)^2=\frac{4\sin^2(\epsilon)}{\sin^2(\epsilon)+\left(\cos(\epsilon)-\frac{1}{2(\cos(\epsilon)+1)}\right)^2}\ \approx\ 0.$$

5.5 Piecewise Smooth Boundaries of Maximum Resistance

In [Pla06] A. Plakhov addresses the problem of *finding piecewise regular non self-intersecting curves or surfaces of maximal resistance*, in the sense that, *seen as mirrors, they reflect light-rays in the exact opposite direction of their incidence.*

We present below a rather elementary direct approach to that problem by means of (nonstandard) Infinitesimal Calculus. As in [Pla06], we use the basic reflection property of the ellipse (Figure 5.7):

rays which hit between the foci are also reflected between the foci.

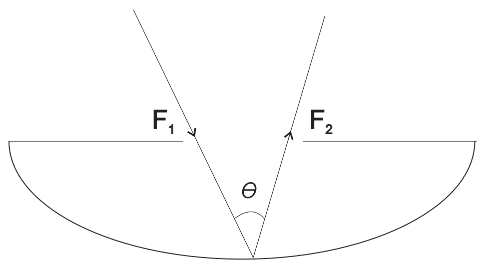

Figure 5.7

In particular, if the ellipse has foci $F_1(-c,0),\ F_2(c,0),$ equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and eccentricity $c/a\approx 0$, then $\theta\approx 0$, *i.e.*, reflection is almost opposite to incidence.

5.5.1 Self-intersecting mirrors

Assume light rays may have any direction whatsoever from above a line segment of length 1 and fix internal sequences $M_i,\ N_i\in{}^*\mathbb{N}_\infty\ (i\in{}^*\mathbb{N})$.

Divide the segment $[0, 1]$ in N_1 equal parts and in each of them define an ellipse with the major axis on the initial segment, as shown in Figure 5.8, where $F_{2i-1,1}$ and $F_{2i,1}$ denote the foci of the $i - th$ ellipse, $i = 1, \ldots, N_1$.

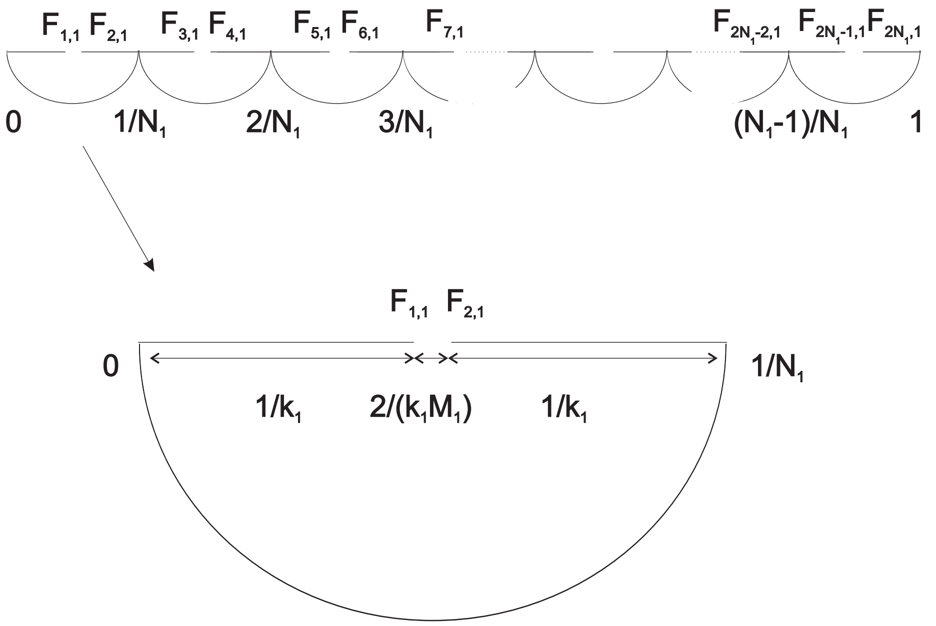

Figure 5.8

Each of the N_1 ellipses verifies the following conditions for exactness of subdivision.

$$k_1 = 2N_1 \left(1 + \frac{1}{M_1}\right),$$

$$a_1 = \frac{1}{k_1} + \frac{1}{k_1 M_1}, \; b_1 = \frac{1}{k_1}\sqrt{1 + \frac{2}{M_1}}, \; c_1 = \frac{1}{k_1 M_1}.$$

Therefore the eccentricity $e_1 \approx 0$ as required. Moreover the probability P_1 that a light ray falls out of the foci window is given by

$$P_1 = N_1 \cdot \frac{2}{k_1} = \frac{M_1}{M_1 + 1} \approx 1.$$

Next define new elipses in each of the segments $[(j-1)/N_1, F_{2j-1,1}]$ and $[F_{2j,1}, j/N_1]$ for $j = 1, \ldots, N_1$. Note that both segments have length $1/k_1$ and divide each of them into N_2 equal parts wherein elipses are defined again with foci $F_{2i-1,2}$ and $F_{2i,2}$, $i = 1, \ldots, N_2$

according to the following conditions

$$k_2 = 2^2 N_1 N_2 \left(1 + \frac{1}{M_1}\right)\left(1 + \frac{1}{M_2}\right),$$

$$a_2 = \frac{1}{k_2} + \frac{1}{k_2 M_2}, \quad b_2 = \frac{1}{k_2}\sqrt{1 + \frac{2}{M_2}}, \quad c_2 = \frac{1}{k_2 M_2}.$$

The probability P_2 that a light ray falls out of the foci windows is given by

$$P_2 = 2N_1 N_2 \frac{2}{k_2} = \left(\frac{M_1}{M_1 + 1}\right)\left(\frac{M_2}{M_2 + 1}\right).$$

Iteration of this procedure follows the pattern

$$k_i = 2^i \prod_{j=1}^{i} N_j \prod_{j=1}^{i} \left(1 + \frac{1}{M_j}\right),$$

$$a_i = \frac{1}{k_i} + \frac{1}{k_i M_i}, \quad b_i = \frac{1}{k_i}\sqrt{1 + \frac{2}{M_i}}, \quad c_i = \frac{1}{k_i M_i}.$$

Interestingly enough, whatever the sequence N_i might be

$$P_i = \prod_{j=1}^{i} \frac{M_j}{M_j + 1}$$

In particular, if for some fixed $N \in {}^*\mathbb{N}_\infty$ all the $M_j = N$, then

$$P_{N^2} = \left(1 - \frac{1}{N+1}\right)^{N^2} \approx e^{-\frac{N^2}{N+1}} \approx 0. \tag{5.14}$$

Assume from now on that for some fixed $N \in {}^*\mathbb{N}_\infty,\ M_j \equiv N$ so that (5.14) holds.

The possibility that a ray entering a foci window hits one of the smaller elipses and is not reflected conveniently must also be considered. The following discusses this situation. Consider Figure 5.9, where one elipse is centered at the origin of coordinates for simplicity.

Let the light ray r pass through the window $[F_{1,i-1}F_{2,i-1}]$ with inclination θ.

As a matter of notational simplification, define

$$A := 2K_{i-1}N_i \ \text{ and } \ B := \frac{K_{i-1}N}{2}.$$

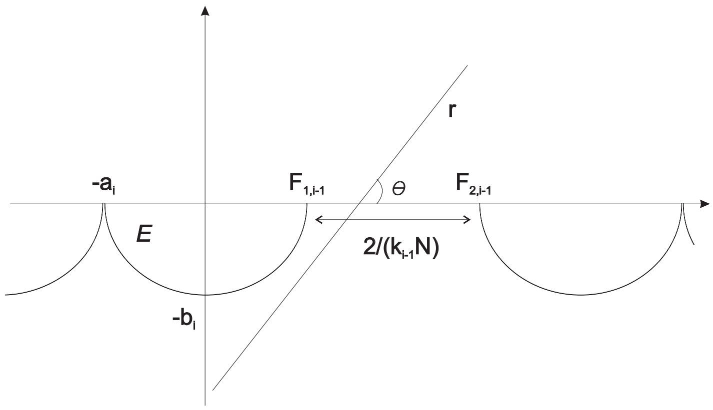

Figure 5.9

The centered ellipse is given by

$$\frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} = 1 \quad \text{with} \quad a_i = \frac{1}{A} \quad \text{and} \quad b_i = \frac{N}{A(N+1)}\sqrt{1+\frac{2}{N}}.$$

An equation of the light ray is

$$y_t = \tan\theta\left(x - \frac{1}{A} - \frac{t}{B}\right) \quad \text{for some } t \in]0,1[.$$

The light ray intersects the ellipse at a point (x, y_t) when

$$\theta = \arctan\left(\frac{\sqrt{1-x^2A^2}}{B+At-ABx} \cdot B \cdot \frac{\sqrt{N(N+2)}}{N+1}\right)$$

necessarily with

$$0 < x < 1/A;$$

but then

$$\begin{aligned} 0 < \theta \quad &< \quad \arctan\left(\frac{B}{At} \cdot \frac{\sqrt{N(N+2)}}{N+1}\right) \\ &= \quad \arctan\left(\frac{N}{N_i t} \cdot \frac{\sqrt{N(N+2)}}{4(N+1)}\right) \end{aligned}$$

therefore $\theta \approx 0$ as long as $\dfrac{N}{N_i t} \approx 0$ and this happens whenever $t \geq \dfrac{1}{N}$ and $N_i = N^3$, thus the probability that the entering light rays hit a smaller ellipse is approximately

$$\sum_{j=1}^{N^2-1} \frac{2^{j+1}}{N^2 k_j} \prod_{i=1}^{j} N_i = \frac{2}{N^2} \sum_{j=1}^{N^2-1} \left(\frac{N}{N+1}\right)^j =$$

$$\frac{2}{N}\left(1-\left(\frac{N}{N+1}\right)^{N^2-1}\right) \approx \frac{2}{N}\left(1-e^{-\frac{N^2-1}{N+1}}\right)$$

hence infinitesimal. Summarizing:

As long as all the $M_i = N$ and $N_i = N^3$, for some $N \in {}^*\mathbb{N}_\infty$, the N^2-th step of the foregoing procedure entails a self-intersecting "mirror" which reflects light rays along directions infinitely near the incidence direction with probability infinitely near 1.

Although self-intersecting, our curve is $^* -$ *continuous* and maximizes resistance.

5.5.2 Simple mirrors

From now on we will take all the $N_i = N^3$.

We eliminate self-intersections "indirectly" as illustrated in Figure 5.10: extend the mirror infinitesimally towards the center of each ellipse $[-c_i, -P] \cup [P, c_i]$, and connect with the ellipse itself by means of two straight line segments r and $\overline{r}$ of adequate inclination θ.

The angle θ must of course be infinitesimal, but also such that the line r, and its symmetric $\overline{r}$, do not intersect any of the inner ellipses. Finally, having thus created more "reflective" regions, their total length must be infinitesimal. We now sketch calculations

$$c_i = \frac{1}{k_i N} \qquad a_{i+1} = \frac{1}{2k_i N^3} \qquad b_{i+1} = \frac{1}{2(N+1)k_i N^2}\sqrt{1+\frac{2}{N}}$$

For some positive ϵ to be determined, the center C of the first inner ellipse and the end point P verify

$$C = c_i + a_{i+1} = \frac{2N^2+1}{2k_i N^3} \qquad P = \frac{c_i}{1+\epsilon}.$$

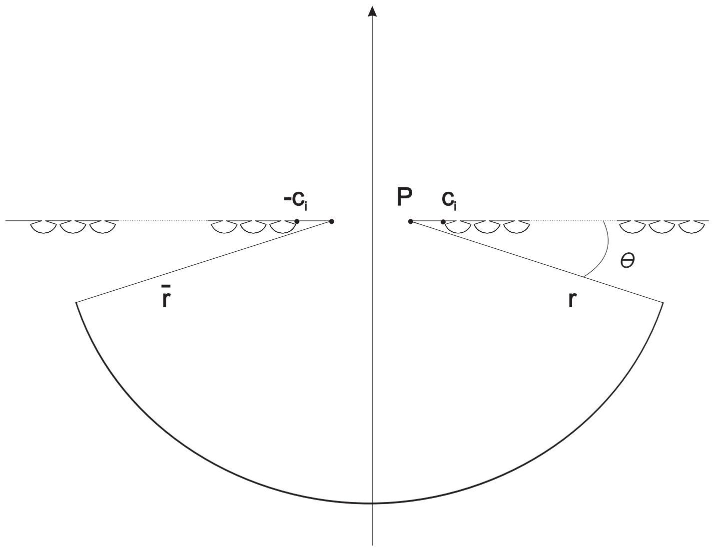

Figure 5.10

The line r and inner ellipse $\mathcal{E}$ satisfy

$$r \equiv y = \tan\theta\,(x-P) \qquad\qquad \mathcal{E} \equiv \frac{(x-C)^2}{a_{i+1}^2} + \frac{y^2}{b_{i+1}^2} = 1$$

The angle τ for which r is tangent to $\mathcal{E}$ is given by

$$\tau = \arctan \frac{-\frac{b_{i+1}}{a_{i+1}}\sqrt{a_{i+1}^2 - (x-C)^2}}{x-P} \quad (c_i < x < C).$$

Now, $\tau \approx 0$ whenever $\dfrac{\sqrt{a_{i+1}^2-(x-C)^2}}{x-P} \approx 0$; but,

$$0 \;\leq\; \frac{\sqrt{a_{i+1}^2-(x-C)^2}}{x-P} \;\leq\; \frac{a_{i+1}}{x-P} \;\leq\; \frac{a_{i+1}}{c_i}\frac{1+\epsilon}{\epsilon} \;\leq\; \frac{1}{N^2\epsilon}$$

and $\tau \approx 0$ when $\epsilon = \frac{1}{N}$. Any infinitesimal angle $\theta > \tau$ may be used to eliminate the self-intersection. Moreover, as

$$c_i - P = \frac{1}{k_i N(N+1)} < \frac{1}{N}\frac{2}{k_i N}$$

the probability of a ray being inadequately reflected by this procedure is infinitesimal.

Summarizing, the probability of a ray being reflected with opposite direction of incidence is given by

$$\widehat{P_{N^2}} \approx 1 - \left(e^{-\frac{N^2}{N+1}} + \frac{2}{N}\left(1 - e^{-\frac{N^2-1}{N+1}}\right)\right) \approx 1.$$

5.5.3 Convex mirrors

As a matter of making terminology more precise, let $\sigma : {}^*[0,1] \to {}^*\mathbb{R}^2$ be the curve thus defined in sub-section 5.5.2.

When one wants to take into account the size and the position of the mirror an affine transformation is in order: given distinct points P and Q in $\mathbb{R}^2$, let

$$\begin{aligned} (v_1, v_2) &:= Q - P \\ M &:= \begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix} \\ \sigma_{PQ}(t) &:= P + M\sigma(t) \quad (t \in {}^*[0,1]); \end{aligned}$$

σ_{PQ} describes the (simple plane) mirror positioned along $\overrightarrow{v}$, which we may re-parametrize in $I := [a,b]$ $(a < b)$ by

$$\sigma^I_{PQ}(t) := \sigma_{PQ}\left(\frac{t-a}{b-a}\right) \qquad (t \in I). \tag{5.15}$$

Suppose now that $\alpha : [0,\ell] \subseteq \mathbb{R} \to \mathbb{R}^2$ is a C^1 regular curve parametrized by arc length[1]. Let the "reflective side" of α be its convex side as illustrated in Figure 5.11.

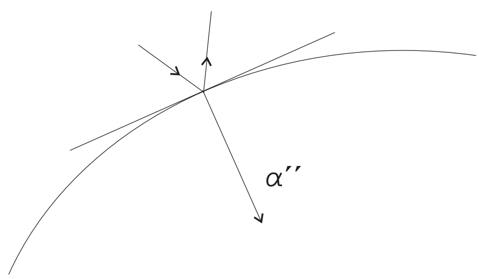

Figure 5.11

[1]Actually it suffices that α is rectifiable so that the following general procedure may be adapted.

A mirror of almost maximum resistance adjusted to the curve may be described the following way

1. Pick an infinite $N \in {}^*\mathbb{N}_\infty$ and define for $0 \leq j \leq 2N$

$$
\begin{aligned}
a_j &:= \begin{cases} \frac{j/2}{N} & \text{if j is even} \\ \frac{(j+1)/2}{N} - \frac{1}{N^2} & \text{if j is odd} \end{cases} \\
b_j &:= \ell a_j
\end{aligned}
$$

so that

$$
\begin{aligned}
[0,\ell] &= \bigcup_{j=1}^{2N} [b_{j-1}, b_j] \\
b_j - b_{j-1} &= \begin{cases} \frac{\ell}{N^2} & j \text{ is even} \\ \frac{\ell}{N} - \frac{\ell}{N^2} & j \text{ is odd} \end{cases} \quad (1 \leq j \leq 2N)
\end{aligned}
$$

2. Define

$$
\begin{aligned}
P_j &:= \alpha(b_j) \qquad (0 \leq j \leq 2N) \\
I_j &:= [b_j, b_{j+1}] \qquad (0 \leq j \leq 2N-1)
\end{aligned}
$$

and consider the polygon $[P_0, P_1, \cdots, P_{2N}]$. Also define (vide (5.15) above)

$$
\mu_j(t) := \begin{cases} \sigma^{I_j}_{P_j P_{j+1}}(t) & \text{if } t \in I_j \ \ \& \ j \text{ is even} \\ \\ P_j + \dfrac{N^2}{\ell}(t-b_j)(P_{j+1}-P_j) & \text{if } t \in I_j \ \ \& \ j \text{ is odd} \end{cases} \qquad (0 \leq j \leq 2N-1)
$$

Finally $\mu_0 + \cdots + \mu_{2N-1}$ is a mirror of almost maximum resistance whose standard part is α. Under infinite magnification, the geometry between P_j and P_{j+2}, with j even, is exemplified in Figure 5.11 below.

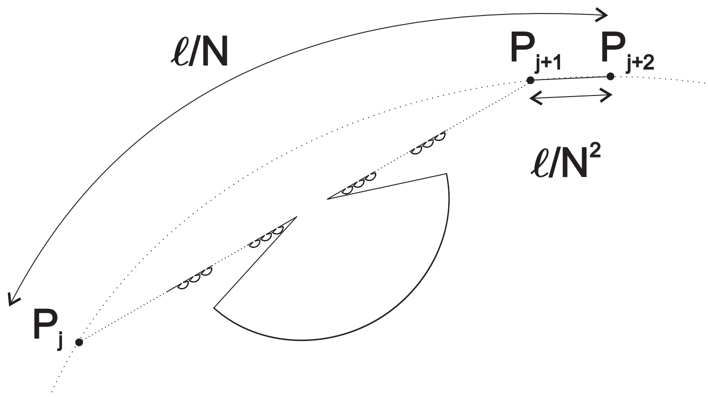

Figure 5.11

5.5.4 Calculus of the resistance

Given any curve α, its resistance is given by the formula

$$R := \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(\varphi - \varphi^+(x, \varphi))) \cos\varphi \, d\varphi \, dx$$

where φ is the angle of incidence and φ^+ the angle of reflection, which depends on φ and the point of entrance x (see Figure 5.12).

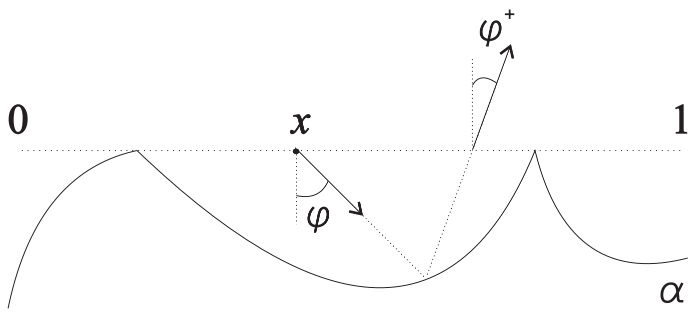

Figure 5.12

For example, if α is a straight line, then $\varphi^+ = -\varphi$, for every x and henceforth $R = 8/3$. We also remark that the maximum resistance of any curve is 4.

We will now evaluate the resistance of the curve obtained in 5.5.2 by minimizing R. To do so, we must maximize the angle $\varphi - \varphi^+$. We assume that the ray light hits one inner ellipse between the foci, so that the direction of the reflected ray is almost inverted (elsewhere the probability is approximately zero). Therefore the angle of reflection $\varphi - \varphi^+$ is less than the angle of reflection when a ray light hits one of the foci (and consequently the ray is reflected in the second foci).

Let us consider the general case (the i-step) and let θ be half of the maximum angle of reflection, as exemplified in Figure 5.13.

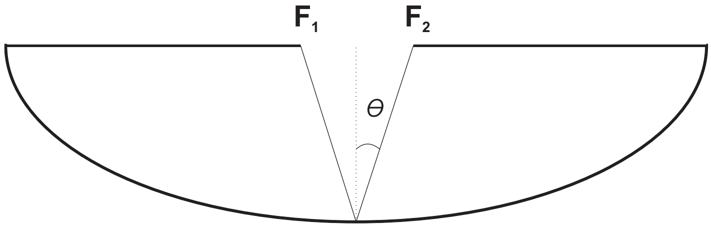

Figure 5.13

Therefore

$$\tan\theta = \frac{c_i}{b_i} = \frac{1}{\sqrt{N(N+2)}}$$

and so

$$\cos(\varphi - \varphi^+(x,\varphi)) \geq \cos\left(2\arctan\frac{1}{\sqrt{N(N+2)}}\right) = 1 - \frac{2}{(N+1)^2}$$

and

$$R \gtrsim \left(2 - \frac{2}{(N+1)^2}\right)\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi\, dx = 4 - \frac{4}{(N+1)^2} \approx 4.$$

5.6 Regular Surfaces

The aim of this section is to present a nonstandard characterization of a regular surface. To start, let us recall the following definition.

Definition 5.10 *Let $S \subseteq \mathbb{R}^3$ be a nonempty set. We say that S is a* ***regular surface*** *if for each $P \in S$, there exist an open neighbourhood V of P, an open set U in $\mathbb{R}^2$ and a function $x : U \to V \cap S$ satisfying the following conditions:*

1. *x is a homeomorphism;*

2. *x is of class C^1;*

3. *for each $q \in U$, the differential $Dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is $1-1$.*

The function x is called a **parametrization** of S in (a neighbourhood of) P.

As usual, we denote $x_u(q) := \dfrac{\partial x}{\partial u}(q)$ and $x_v(q) := \dfrac{\partial x}{\partial v}(q)$.

Definition 5.11 *If $x : U \to V \cap S$ is a parametrization in $P = x(p)$, we define the **unit normal vector** at each point $Q = x(q) \in x(U)$ by the rule*

$$N(Q) := \frac{x_u \times x_v}{|x_u \times x_v|}(q).$$

Since x is a C^1 function, N is continuous.

[HJ01] contains a nonstandard characterization of submanifolds in Euclidean spaces, which we will use in order to give a characterization of regular surfaces by means of a field of unit normal vectors on the set.

Theorem 5.12 *[HJ01] A standard subset $M^m \subseteq \mathbb{R}^n$ with $n \in {}^\sigma\mathbb{N}$ is a C^1-submanifold if and only if there exists a standard tangent plane function $T : M \to G(m,n)$ into the set of affine m-planes such that, for every $P \in ns({}^*M)$,*

1. *$P \in T(P)$;*

2. *the ortogonal projection $\pi_P : {}^*M \to T(P)$ is an infinitesimal bijection in the sense that*

   (a) *if $R, R' \in {}^*M$ with $R \approx R' \approx P$ and $\pi_P(R) = \pi_P(R')$, then $R = R'$;*

   (b) *if $Q \in T(P)$ and $Q \approx P$, then there exists $R \in {}^*M$ with $R \approx P$ and $\pi_p(R) = Q$;*

3. *if ${}^*M \ni Q \approx P$ then $\dfrac{|Q - \pi_P(Q)|}{|Q-P|} \approx 0$, i.e., the angle between the secant line through P and Q and the plane $T(P)$ is infinitesimal.*

We present now our result:

Theorem 5.13 *[Alm07a] Let $S \subseteq \mathbb{R}^3$ be a nonempty set. Then S is a regular surface if and only if for each $P \in ns({}^*S)$, there exist a standard neighbourhood ${}^*V$ of P and a standard continuous function $N : V \cap S \to \mathbb{R}^3$ such that:*

1. *for all $Q \in V \cap S$, $|N(Q)| = 1$;*

2. *for all $Q, R \in ns({}^*V \cap {}^*S)$ with $Q \neq R$,*

$$R \approx Q \Rightarrow N(Q) \cdot \frac{Q - R}{|Q - R|} \approx 0;$$

3. *If $T(P)$ is the plane containing P and orthogonal to $N(P)$, then*

$$\mu(P) \cap T(P) \subseteq \pi_P(\mu(P) \cap {}^*S)$$

where $\pi_P : {}^*\mathbb{R}^3 \to T(P)$ is the orthogonal projection.

Proof. We begin by assuming that S is a regular surface and let us fix $P \in ns({}^*S)$. Choose a standard neighbourhood V of $st(P)$ and a parametrization $x : U \to V \cap S$ in P. Define $N : V \cap S \to \mathbb{R}^3$ as the unit normal vector function at $x(U)$. It is easy to see that conditions 1 and 2 are satisfied. About condition 3, observe that $T(P)$ is the tangent plane to the surface at P, and by Theorem 5.12, condition 2, the proof follows.

To prove the reverse, we will prove that there exists a standard function $T : S \to G(2, 3)$, (where $G(2, 3)$ denotes the set of planes in $\mathbb{R}^3$) such that, for each $P \in ns({}^*S)$, we have:

1. $P \in T(P)$;

2. the orthogonal projection $\pi_P : {}^*S \to T(P)$ is an infinitesimal bijection;

3. If ${}^*S \ni Q \approx P$ then $\dfrac{|Q - \pi_P(Q)|}{|Q - P|} \approx 0$.

Since this is a local problem, we will define a standard function $T : V \cap S \to G(2, 3)$, where ${}^*V$ is a neighbourhood of P. First, choose a continuous function $u_1 : V \cap S \to \mathbb{R}^3$ such that $u_1(Q) \cdot N(Q) = 0$ and $|u_1(Q)| = 1$, for all $Q \in V \cap S$. Define $u_2 : V \cap S \to \mathbb{R}^3$ by the rule

$u_2(Q) := u_1(Q) \times N(Q)$ and let

$$\begin{array}{rcl} T: \ V \cap S & \to & G(2,3) \\ Q & \mapsto & \{Q + \lambda_1 u_1(Q) + \lambda_2 u_2(Q) \mid \lambda_1, \lambda_2 \in \mathbb{R}\} \end{array}$$

Clearly, $P \in T(P)$.

Suppose now that there exist $R, R' \in {}^*S$ with $R \approx R' \approx P$ and $\pi_P(R) = \pi_P(R')$ but $R \neq R'$. Thus

$$P + ((R-P) \cdot u_1(P)) \cdot u_1(P) + ((R-P) \cdot u_2(P)) \cdot u_2(P)$$

$$= P + ((R'-P) \cdot u_1(P)).u_1(P) + ((R'-P) \cdot u_2(P)) \cdot u_2(P)$$

$$\Leftrightarrow \begin{cases} (R-R') \cdot u_1(P) = 0 \\ (R-R') \cdot u_2(P) = 0 \end{cases}.$$

So we may conclude that

$$\frac{R-R'}{|R-R'|} = \pm N(P).$$

Multiplying both members by $N(R)$, we get

$$N(R) \cdot \frac{R-R'}{|R-R'|} = \pm N(R) \cdot N(P).$$

Moreover, the first member of this equation is infinitesimal and the second member is infinitely close to ± 1 (a contradiction). So the function is $1-1$. The onto condition follows from 3.

Finally, the angle between the plane $T(P)$ and the straight line PQ is infinitesimal because

$$N(P) \cdot \frac{Q-P}{|Q-P|} \approx 0$$

and $N(P)$ is orthogonal to $T(P)$.

■

Let us note that it is also true that

$$\pi_P(\mu(P) \cap {}^*S) \subseteq \mu(P) \cap T(P)$$

because if $Q \in {}^*S$ with $Q \approx P$, the continuity of π_P implies that

$$\pi_P(Q) \approx \pi_P(P) = P \in T(P).$$

Chapter 6

Differentiable Manifolds

In this chapter we develop an analog of the classical theory of differentiable manifolds, formulated in terms of nonstandard analysis. Many of the classical concepts that we deal with can be presented using a kind of internal functions, which we will call δ-infinitesimal transformations. The idea is that these functions move infinitely nearstandard points of the manifold, with some smoothness properties.

In [Sch97], the author presents a nonstandard manifold theory. He considers two type of manifolds, what he called *Concrete* and *Abstract m-manifolds*. These are internal sets where the transition functions are S-continuous, m-differentiable and the derivative operator is S-continuous. So he uses nonstandard methods to generalize the manifold concept. We will stay inside the category of classical manifold theory, working with standard manifolds, and use nonstandard methods to present new definitions like Tangent Space, Derivative of a standard function, etc.

6.1 Tangent Space to a Differentiable Manifold

For the sake of completeness, let us recall the definition of a differentiable manifold modeled on an arbitrary real Banach space E (*vide* any of [AMR83], [CBDM77], [Ish99], [Lan95] and

[Lee03]).

Definition 6.1 *Let M be a nonempty Hausdorff topological space and $\{(U_i, x_i)\}$ $(i \in I)$ a family of pairs satisfying the following conditions:*

1. *Each U_i is an open subset of M and $x_i : U_i \to x_i(U_i) \subseteq E$ is a homeomorphism;*
2. *The U_i cover M: $\cup_{i\in I} U_i = M$;*
3. *When $U_i \cap U_j \neq \emptyset$, the function $x_i x_j^{-1} : x_j(U_i \cap U_j) \to x_i(U_i \cap U_j)$ is of class C^k;*
4. *The set $\{(U_i, x_i)_{i\in I}\}$ is maximal for the previous conditions, i.e., the set contains all functions with these properties;*

*then we say that M is a **Differentiable Manifold** of class C^k. When $k = \infty$ the manifold is called **smooth**. If $dim(E) = n \in \mathbb{N}$ we say that M is a n-dimensional manifold.*

The pair (U_i, x_i) is called a **chart** and $x_i x_j^{-1}$ the **transition** or **overlap function**; we say that the functions x_i and x_j are **smoothly compatible** . If a point p of M lies in U_i, then we say that (U_i, x_i) is a chart at p. The family of functions $\mathcal{A} := \{(U_i, x_i) \,|\, i \in I\}$ is an **atlas** on M. Observe that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

For $p \in {}^{\sigma}M$ and $q \in {}^{*}M$, we say that

$$p \approx q \Leftrightarrow q \in \mu(p) := \bigcap \{{}^{*}O \,|\, p \in O \text{ and } O \text{ is open.}\}$$

Given a manifold M we can describe the tangent space to M using a type of functions defined on M, the δ-infinitesimal transformations. The set obtained is a linear space isomorphic to the real Banach space. From now on we will assume that M is a differentiable manifold of class C^k with $k \geq 2$.

Definition 6.2 *Let δ be a fixed positive infinitesimal and $p \in M$. Let $A \subseteq {}^*M$ be an internal set, $X : A \to X(A) \subseteq {}^*M$ an internal bijection such that*

- $\mu(p) \subseteq A \cap X(A)$*;*
- $X(q) \approx q$ *for all* $q \in ns(A)$*,*
- $X^{-1}(q) \approx q$ *for all* $q \in ns(X(A))$*.*

*We say that X is a δ-**infinitesimal transformation** at p if there exists a chart (U, x) with $\mu(p) \subseteq {}^*U \subseteq A$ such that*

$$\overline{X}(u) := \frac{xXx^{-1}(u) - u}{\delta} \quad \text{and} \quad \overline{X^{-1}}(u) := \frac{xX^{-1}x^{-1}(u) - u}{\delta}$$

are both SU-differentiable at

$$C := \{u \in {}^*x(U) \,|\, Xx^{-1}(u) \in {}^*U \wedge X^{-1}x^{-1}(u) \in {}^*U\}.$$

The set of all δ-infinitesimal transformations on M at p will be denoted by $\delta\Theta_p M$.

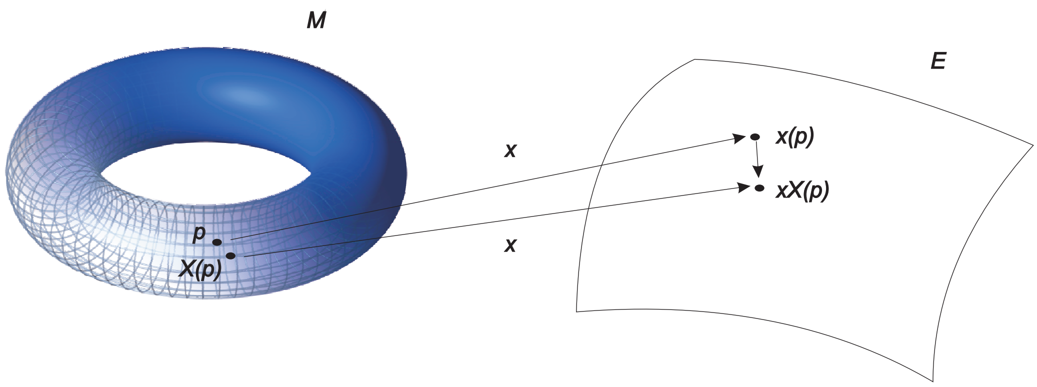

Figure 6.1

The set C contains all nearstandard points of ${}^*x(U)$. In fact, if $u \in ns({}^*x(U))$, that is, $x^{-1}(u) \in ns({}^*U)$, then

$$Xx^{-1}(u) \approx x^{-1}(u) \approx st(x^{-1}(u)) \in {}^\sigma U.$$

As U is open, it follows that $Xx^{-1}(u) \in {}^*U$. In a similar way for X^{-1}, one proves the desired.

As for notation simplification, in what follows, $\underline{X}(u) := xXx^{-1}(u)$. So

$$\overline{X}(u) = \frac{\underline{X}(u) - u}{\delta} \Leftrightarrow \underline{X}(u) = \delta\overline{X}(u) + u.$$

Consequently $\underline{X}$ is also SU-differentiable and $D\underline{X}_u = \delta D\overline{X}_u + I$. The same argument for $\underline{X}^{-1}$.

For example, let $\mathcal{S}^2 \subset \mathbb{R}^3$ be the 2-dimensional sphere and X be the function given by

$$X(\cos a \sin b, \sin a \sin b, \cos b) = (\cos(a+\epsilon)\sin(b+\eta), \sin(a+\epsilon)\sin(b+\eta), \cos(b+\eta)),$$

$$a \in {}^*\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[, b \in {}^*\left]\frac{\pi}{4}, \frac{3\pi}{4}\right[, \epsilon, \eta \approx 0 \text{ with } \frac{\epsilon}{\delta}, \frac{\eta}{\delta} \in fin({}^*\mathbb{R})$$

where δ is a fixed positive infinitesimal.

Let $p = (\cos\pi\sin(\pi/2), \sin\pi\sin(\pi/2), \cos(\pi/2)) = (-1,0,0) \in \mathcal{S}^2$ and x be the chart on $U = \{(x_1,x_2,x_3) \in \mathcal{S}^2 \mid a \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[, b \in \left]\frac{\pi}{4}, \frac{3\pi}{4}\right[\}$ given by $x(x_1,x_2,x_3) := (x_2,x_3)$.

$$\begin{aligned}
\overline{X}(u) &= \frac{(\sin(a+\epsilon)\sin(b+\eta) - \sin(a)\sin(b), \cos(b+\eta) - \cos(b))}{\delta} \\
&= \left(\frac{\sin(a+\epsilon) - \sin(a)}{\delta}\sin(b+\eta) + \frac{\sin(b+\eta) - \sin(b)}{\delta}\sin(a),\right. \\
&\left.\frac{\cos(b+\eta) - \cos(b)}{\delta}\right) \\
&= \left(\frac{\epsilon}{\delta}\frac{\sin(a+\epsilon) - \sin(a)}{\epsilon}\sin(b+\eta) + \frac{\eta}{\delta}\frac{\sin(b+\eta) - \sin(b)}{\eta}\sin(a),\right. \\
&\left.\frac{\eta}{\delta}\frac{\cos(b+\eta) - \cos(b)}{\eta}\right) \\
&\approx \left(\frac{\epsilon}{\delta}\cos(a)\sin(b) + \frac{\eta}{\delta}\cos(b)\sin(a), -\frac{\eta}{\delta}\sin(b)\right).
\end{aligned}$$

For $a = \pi$ and $b = \pi/2$, we get

$$st(\overline{X}x(p)) = \left(st\left(-\frac{\epsilon}{\delta}\right), st\left(-\frac{\eta}{\delta}\right)\right).$$

For example, if $\epsilon = 0$ and $\eta = \pm\delta$ then $st(\overline{X}x(p)) = (0, \mp 1)$ and for $\epsilon = \pm\delta$ and $\eta = 0$, $st(\overline{X}x(p)) = (\mp 1, 0)$.

We note that the previous definition is independent of the chart: if $\overline{X}$ and $\overline{X^{-1}}$ are SU-differentiable at u for a certain chart (U,x), the same occurs for every other chart (V,y), because

$$\overline{X}_1(u) := \frac{yXy^{-1}(u) - u}{\delta} = \frac{(yx^{-1})\underline{X}(xy^{-1})(u) - (yx^{-1})(xy^{-1})(u)}{\delta}$$

where $\underline{X} = xXx^{-1}$, and so, for some infinitesimal function $\eta(\cdot)$ [1]:

$$D(\overline{X}_1)_u =$$

$$= \frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\underline{X}_{(xy^{-1})(u)} D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$

$$= \frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} (\delta D\overline{X}_{(xy^{-1})(u)} + I) D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$

$$= D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\overline{X}_{(xy^{-1})(u)} D(xy^{-1})_u +$$

$$\frac{D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D(xy^{-1})_u - D(yx^{-1})_{(xy^{-1})(u)} D(xy^{-1})_u}{\delta}$$

$$= D(yx^{-1})_{\underline{X}(xy^{-1})(u)} D\overline{X}_{(xy^{-1})(u)} D(xy^{-1})_u +$$

$$\left[D^2(yx^{-1})_{(xy^{-1})(u)}(\overline{X}(xy^{-1})(u), \cdot) + |\overline{X}(xy^{-1})(u)| \eta(\cdot)\right] D(xy^{-1})_u$$

which is finite. It can be proven analogously that $\overline{X_1^{-1}}$ is also SU-differentiable.

Theorem 6.3 *Let $p \in M$ be a point and let ${}^*M \ni a \approx p$. Then there exist $\delta \approx 0$ and $X \in \delta\Theta_p M$ such that $X(p) = a$.*

Proof. Let (U, x) be a chart at p and define

$$\begin{aligned} \delta :=&\ \ |x(a) - x(p)| \approx 0; \\ X(q) :=&\ \ x^{-1}(x(q) + x(a) - x(p)). \end{aligned}$$

Then X is invertible with inverse $X^{-1}(r) := x^{-1}(x(r) - x(a) + x(p))$. Moreover,

$$\overline{X}(u) = \frac{xXx^{-1}(u) - u}{\delta} = \frac{x(a) - x(p)}{\delta}$$

and $\overline{X^{-1}}$ are SU-differentiable. Consequently, $X \in \delta\Theta_p M$ and $X(p) = a$. ■

Also recall:

Definition 6.4 *Let M and N be two differentiable manifolds. A function $f : M \to N$ is of class C^k if for each $p \in M$, and a chart (U, x) in M with $p \in U$ and a chart (V, y) with $f(U) \subseteq V$, the composite function $yfx^{-1} : x(U) \to y(V)$ is a C^k function.*

[1] in the sense that $\eta(x) \approx 0$ whenever $x \in fin({}^*E)$

Therefore a function $f : M \to \mathbb{R}$ is a C^k-function if and only if for every $p \in M$ there is some chart (U, x) at p so that $fx^{-1} : x(U) \to \mathbb{R}$ is a C^k-function.

Theorem 6.5 *Let $f : M \to \mathbb{R}$ be a function. Then f is of class C^1 if and only if for all $p \in ns(^*M)$ there exists an internal finite linear operator $L_p \in {}^*L(E, \mathbb{R})$ such that*

$$\forall 0 \approx \delta \in {}^*\mathbb{R}^+ \, \forall X \in \delta\Theta_{st(p)}M \quad f(Xp) - f(p) = L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta \quad (\eta \approx 0).$$

Proof. If $p \in ns(^*M)$ then $x(p) \in ns(^*E)$. By hypothesis, fx^{-1} is a C^1 function. Hence $D(fx^{-1})_{x(p)}$ exists in ${}^*L(E, \mathbb{R})$ and is a finite linear operator. Define $L_p := D(fx^{-1})_{x(p)}$. Fix now a positive $\delta \approx 0$ and $X \in \delta\Theta_{st(p)}M$. Then

$$\begin{aligned} f(X(p)) - f(p) &= (fx^{-1})(xX(p)) - (fx^{-1})(x(p)) \\ &= L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta \qquad (\eta \approx 0). \end{aligned}$$

To prove the converse, let us see that fx^{-1} is differentiable at $x(p) \in ns(^*E)$, *i.e.*, there exists a finite linear operator $L' \in {}^*L(E, \mathbb{R})$ such that for all $0 \approx \epsilon \in {}^*E$,

$$(fx^{-1})(x(p) + \epsilon) - (fx^{-1})(x(p)) = L'(\epsilon) + |\epsilon|\eta,$$

for some infinitesimal η. To begin with, since $x(p) \in ns(^*E)$ then $p \in ns(^*M)$. Define $L' := L_p$ and fix any $\epsilon \approx 0$. Let $\delta := |\epsilon| \approx 0 \in {}^*\mathbb{R}^+$ (when $\epsilon = 0$ it is obvious). Define now

$$X(q) := x^{-1}(x(q) + \epsilon).$$

We will prove that $X \in \delta\Theta_{st(p)}M$: we have that X is an internal bijection with inverse

$$X^{-1}(r) = x^{-1}(x(r) - \epsilon).$$

Besides this, $\overline{X}(u) = \epsilon/\delta = \epsilon/|\epsilon|$ and $\overline{X^{-1}}(u) = -\epsilon/|\epsilon|$ are both SU-differentiable. As a result

$$\begin{aligned} fx^{-1}(x(p) + \epsilon) - fx^{-1}x(p) &= f(X(p)) - f(p) \\ &= L_p(xX(p) - x(p)) + |xX(p) - x(p)|\eta \\ &= L'(\epsilon) + |\epsilon|\eta. \end{aligned}$$

∎

For $X, Y \in \delta\Theta_p M$, we say that they are δ-equivalent at $x(p)$ if $st(\overline{X}x(p)) = st(\overline{Y}x(p))$ and we write $X \equiv_{x(p)} Y$, or $X \equiv Y$ if there is no danger of confusion.

The set $\delta\Theta_p M$ forms a group under composition of functions. Although the operation is not commutative we have the following approximation: for $u \approx x(p)$,

$$
\begin{aligned}
\overline{XY}(u) &= \frac{\underline{X}\,\underline{Y}(u) - u}{\delta} \\
&= \frac{\underline{X}\,\underline{Y}(u) - \underline{Y}(u)}{\delta} + \frac{\underline{Y}(u) - u}{\delta} \\
&= \overline{X}(\underline{Y}(u)) + \overline{Y}(u) \\
&\approx \overline{X}(u) + \overline{Y}(u)
\end{aligned}
$$

because of the S-continuity of $\overline{X}$. This implies that

$$\overline{XY}(u) \approx \overline{YX}(u).$$

Moreover

$$\overline{X^{-1}}(u) \approx -\overline{X}(u)$$

since

$$0 = \overline{I}(u) = \overline{XX^{-1}}(u) \approx \overline{X}(u) + \overline{X^{-1}}(u).$$

Theorem 6.6 $(\delta\Theta_p M, \circ)$ *is a group.*

Proof. The proof of the theorem is identical to the one in [SL76] with the adequate adjustments.

To see that composition is well defined, take $X, Y \in \delta\Theta_p M$ with $X : A \to X(A)$ and $Y : B \to Y(B)$. Define

$$C := \{b \in B \,|\, Y(b) \in A\}.$$

The set C is internal and contains $\mu(p)$ (because $\mu(p) \subseteq B$ and, for $b \in \mu(p)$, $Y(b) \approx b \approx p$, it is also true that $Y(b) \in A$).

By the Cauchy's Principle (see [SL76], Theorem 8.1.4, pag. 196) there exists an open set W with $\mu(p) \subseteq {}^*W \subseteq C$. Define then $XY : {}^*W \to XY({}^*W)$.

It is also true that $\mu(p) \subseteq XY({}^*W)$ since, if we fix $a \in \mu(p)$, $Y^{-1}X^{-1}(a) \approx a \approx p$ will imply that $Y^{-1}X^{-1}(a) \in {}^*W$ and $XY(Y^{-1}X^{-1})(a) = a \in XY({}^*W)$.

Clearly XY is an internal bijection and

$$\begin{aligned} D\overline{XY}_u &= \frac{D\underline{X}_{\underline{Y}(u)} D\underline{Y}_u - I}{\delta} \\ &= \frac{(\delta D\overline{X}_{\underline{Y}(u)} + I)(\delta D\overline{Y}_u + I) - I}{\delta} \\ &\approx D\overline{X}_{\underline{Y}(u)} + D\overline{Y}_u \end{aligned} \tag{6.1}$$

which is a finite operator. In conclusion $\overline{XY}$ is SU-differentiable. Similarly $\overline{(XY)^{-1}}$ is also SU-differentiable.

It is clear that the composition is associative, $I : {}^*M \to {}^*M$ is the identity element and $X^{-1} \in \delta\Theta_p M$. ■

Remark: Since, by (6.1)

$$0 = D\overline{XX^{-1}}_u \approx D\overline{X}_{\underline{X}^{-1}(u)} + D\overline{X^{-1}}_u$$

it follows that

$$D\overline{X^{-1}}_u \approx -D\overline{X}_{\underline{X}^{-1}(u)}.$$

We can define **sum** and **scalar multiplication** on $\delta\Theta_p M$ in the following way:

For $X, Y \in \delta\Theta_p M$ and $a \in \mathbb{R}$:

$$(X+Y)(q) := XY(q) \;\text{ and }\; aX(q) := x^{-1}(x(q) + a\delta\overline{X}x(p)).$$

Note that it is still true that

$$(X+Y)(q) = x^{-1}(x(q) + \delta\overline{XY}x(q)).$$

By Theorem 6.6, the sum is an internal operation. About the scalar multiplication let $Y(q) := x^{-1}(x(q) + a\delta\overline{X}x(p))$. Then Y is 1-1 with inverse $Y^{-1}(r) = x^{-1}(x(r) - a\delta\overline{X}x(p))$. Besides this, $\overline{Y}(u) = a\overline{X}x(p)$ and $\overline{Y^{-1}}(u) = -a\overline{X}x(p)$ are SU-differentiable at $u \approx x(q)$. To sum up, $aX \in \delta\Theta_p M$.

Now we may define tangent vectors on a manifold.

Definition 6.7 *For $p \in M$ and (U, x) a chart at p, we define the δ-tangent space of M at p as*

$$\delta T_p M := \{(p, st(\overline{X}x(p))) \mid X \in \delta\Theta_p M\}$$

and $(p, st(\overline{X}x(p)))$ is called a ***tangent vector*** *on M at p.*

We say that $(p, st(\overline{X}x(p))) \equiv (p, st(\overline{Y}x(p)))$ if $X \equiv_{x(p)} Y$. The ***tangent space*** *to the manifold at p is*

$$T_p M := \delta T_p M / \equiv$$

and the ***tangent bundle*** *of M is given by the disjoint union*

$$TM := \bigcup_{p \in M} T_p M.$$

This definition of tangent vectors has a number of advantages: it makes the local nature of the tangent space clearer, without requiring the use of bump functions, and it is very intuitive. But it also has an inconvenient: it depends on the choice of charts; nevertheless:

Theorem 6.8 *The set $T_p M$ does not depend on the choice of the infinitesimal δ.*

Proof. Let δ and β be two positive infinitesimal numbers and fix $X \in \delta\Theta_p M$.

Define Y as being

$$\begin{aligned} Y(q) &:= x^{-1}\left(x(q) + \beta\overline{X}x(p)\right) \\ &= x^{-1}\left(x(q) + \beta\frac{xX(p) - x(p)}{\delta}\right). \end{aligned}$$

It is clear that Y is 1-1 with inverse

$$Y^{-1}(r) = x^{-1}\left(x(r) - \beta\overline{X}x(p)\right).$$

Besides this,

$$\overline{Y}(u) := \frac{xYx^{-1}(u) - u}{\beta} = \overline{X}x(p),$$

which is SU-differentiable. Similarly we can prove that $\overline{Y^{-1}}$ is also SU-differentiable. So, in conclusion, $Y \in \beta\Theta_p M$.

Since $\overline{X}x(p) = \overline{Y}x(p)$ then

$$\delta T_p M/\equiv \;=\; \beta T_p M/\equiv$$

■

If we define sum and scalar multiplication on T_pM by

$$(p, st(\overline{X}x(p))) + (p, st(\overline{Y}x(p))) := (p, st(\overline{XY}x(p)))$$

and

$$a(p, st(\overline{X}x(p))) := (p, st(\overline{aX}x(p))),$$

it follows that the set T_pM is a linear space, where $(p, 0) = (p, st(\overline{I}x(p)))$ is the identity element. Observe that we also have

$$(p, st(\overline{X}x(p))) + (p, st(\overline{Y}x(p))) = (p, st(\overline{X}x(p)) + st(\overline{Y}x(p)))$$

and

$$a(p, st(\overline{X}x(p))) = (p, a \cdot st(\overline{X}x(p))).$$

Theorem 6.9 *There exists an isomorphism between T_pM and E.*

Proof. Consider the function Φ_p defined by

$$\begin{array}{rccl} \Phi_p : & T_pM & \to & E \\ & (p, st(\overline{X}x(p))) & \mapsto & st(\overline{X}x(p)) \end{array}$$

Clearly Φ_p is 1-1. Fix now $u \in E$ and let $X(q) := x^{-1}(x(q) + \delta u)$. The function X is invertible with inverse $X^{-1}(r) = x^{-1}(x(r) - \delta u)$. Once $\overline{X}(v) = u$ and $\overline{X^{-1}}(v) = -u$ it follows that $X \in \delta\Theta_p M$.

Furthermore $\Phi_p(p, st(\overline{X}x(p))) = u$ and so Φ_p is also onto.

Finally the operator is linear as can easily be seen. ∎

In the classical literature we may find several definitions of tangent space. We are going to present a brief presentation of one of those.

A tangent vector at p is an equivalence class of C^k paths $\alpha :]-\epsilon, \epsilon[\to M$, with $\alpha(0) = p$ where two paths $\alpha_1 :]-\epsilon, \epsilon[\to M$ and $\alpha_2 :]-\epsilon, \epsilon[\to M$ are called equivalent, $\alpha_1 \equiv_1 \alpha_2$, if $(x\alpha_1)'(0) = (x\alpha_2)'(0)$ for some (and hence for any) chart (U, x) on M with $p \in U$. The tangent space of M at p is the set of all tangent vectors at p, $\Gamma/\equiv_1$, where Γ denotes the set of paths with $\alpha(0) = p$. If we define sum and scalar multiplication by

$$\begin{aligned} (\alpha+\beta)(t) &:= x^{-1}(x(p) + t((x\alpha)'(0) + (x\beta)'(0))) \\ (a\alpha)(t) &:= \alpha(at) \end{aligned}$$

it follows that the tangent space has a linear structure.

Theorem 6.10 *The sets T_pM and $\Gamma/\equiv_1$ are isomorphic.*

Proof. Let

$$\begin{aligned} \Phi: \quad \Gamma/\equiv_1 &\to T_pM \\ \alpha &\mapsto (p, (x\alpha)'(0)) \end{aligned}$$

The δ-infinitesimal transformation associated to α in T_pM is $X(q) := x^{-1}(x(q) + \delta(x\alpha)'(0)) \in \delta\Theta_pM$. The operator Φ is well defined since for $\alpha \equiv_1 \beta$, $\Phi(\alpha) = \Phi(\beta)$.

Φ is a linear operator because for $\alpha, \beta \in \Gamma(x)/\equiv_1$ and $a \in \mathbb{R}$,

$$\begin{aligned} \Phi(\alpha+\beta) &= \Phi(x^{-1}(x(p) + t((x\alpha)'(0) + (x\beta)'(0)))) \\ &= (p, (x\alpha)'(0) + (x\beta)'(0)) \\ &= \Phi(\alpha) + \Phi(\beta) \end{aligned}$$

and

$$\begin{aligned} \Phi(a\alpha) &= \Phi(\alpha(at)) \\ &= (p, a(x\alpha)'(0)) \\ &= a\Phi(\alpha) \end{aligned}$$

Clearly Φ is 1-1. To prove that is also onto let

$$\Phi^{-1}(p, st(\overline{X}(p))) := x^{-1}(x(p) + t\, st(\overline{X}x(p))), \text{ for } (p, st(\overline{X}x(p))) \in T_pM.$$

The curve $t \mapsto x^{-1}(x(p) + t\, st(\overline{X}x(p)))$ is well defined in a neighbourhood of zero since $x(U)$ is an open set.

In addition,

$$\Phi\Phi^{-1}(p, st(\overline{X}x(p))) = (p, st(\overline{X}x(p)))$$

and

$$\Phi^{-1}\Phi(\alpha) = x^{-1}(x(p) + t(x\alpha)'(0)) \equiv_1 \alpha(t),$$

as desired. ∎

Apart from this, if $\alpha(t) := x^{-1}(x(p) + t\, st(\overline{X}x(p)))$ then

$$(x\alpha)'(0) = st\frac{x\alpha(\delta) - x\alpha(0)}{\delta} = st(\overline{X}x(p)).$$

Moreover, if

$$(x\beta)'(0) = st(\overline{Y}x(p))$$

then $\alpha \equiv_1 \beta$ if and only if $X \equiv Y$.

This tangent bundle is a smooth manifold in its own right. Let (U, x) and (V, y) be two charts at $p \in U \cap V$ and $X \in \delta\Theta_pM$.

Once the overlap function yx^{-1} is of class C^1, there exists $\eta \approx 0$ such that

$$\frac{yX(p) - y(p)}{\delta} = \frac{(yx^{-1})xX(p) - (yx^{-1})x(p)}{\delta}$$
$$= D(yx^{-1})_{x(p)}\frac{xX(p) - x(p)}{\delta} + \left|\frac{xX(p) - x(p)}{\delta}\right|\eta.$$

If we take the standard part of both members of the last equation one gets

$$st(\overline{X}y(p)) = D(yx^{-1})_{x(p)}st(\overline{X}x(p)).$$

Definition 6.11 *For $p \in M$ and (U, x) a chart at p, let*

$$\tilde{U} := \{(p, st(\overline{X}x(p))) \mid p \in U \wedge X \in \delta\Theta_p M\}$$

and

$$\begin{array}{rccl} \tilde{x}: & \tilde{U} & \rightarrow & E^2 \\ & (p, st(\overline{X}x(p))) & \mapsto & (x(p), st(\overline{X}x(p))) \end{array}$$

The function $\tilde{x}$ is 1-1 because of the 1-1 condition of x and also by the δ-equivalent definition on $\delta\Theta_p M$. Moreover, $\tilde{x}(\tilde{U}) = x(U) \times E$.

Theorem 6.12 *Let M be a differentiable manifold and $\{(U_i, x_i)\}$ $(i \in I)$ an atlas on M. Then $\{(\tilde{U}_i, \tilde{x}_i)\}$ $(i \in I)$ is an atlas on TM. Furthermore, if M is a n-dimensional manifold then TM is a $2n$-dimensional manifold.*

Proof. Simply note that

$$\begin{array}{rccl} \tilde{y}\tilde{x}^{-1}: & \tilde{x}(\tilde{U} \cap \tilde{V}) & \rightarrow & \tilde{y}(\tilde{U} \cap \tilde{V}) \\ & (v, st(\overline{X}x(p))) & \mapsto & (yx^{-1}(v), st(\overline{X}y(p))) \end{array}$$

is differentiable. ■

6.2 Stationary Transformations

Definition 6.13 *Let $X \in \delta\Theta_p M$. We say that X is a* ***stationary transformation*** *at p if $\overline{X}x(p) \approx 0$ for some chart (U, x) at $p \in U$. The set of all stationary transformations at p will be denoted by $\delta I_p M$.*

When x and y are charts at p,

$$\begin{aligned}\overline{X}y(p) &= \frac{yX(p) - y(p)}{\delta} \\ &= \frac{(yx^{-1})xX(p) - (yx^{-1})x(p)}{\delta} \\ &= D(yx^{-1})_{x(p)}\overline{X}x(p) + |\overline{X}x(p)|\eta\end{aligned}$$

for some infinitesimal η, it follows that

$$\overline{X}x(p) \approx 0 \Leftrightarrow \overline{X}y(p) \approx 0,$$

i.e., the definition does not depend on the choice of charts.

Theorem 6.14 *The set δI_pM is a subgroup of $\delta\Theta_pM$.*

Proof. Since

$$\overline{XY}x(p) \approx \overline{X}x(p) + \overline{Y}x(p) \approx 0 \text{ if } \overline{X}x(p) \approx \overline{Y}x(p) \approx 0$$

we proved that $XY \in \delta I_pM$ if $X, Y \in \delta I_pM$. It is clear that $I \in \delta I_pM$ and, given $X \in \delta I_pM$, $X^{-1} \in \delta I_pM$ because

$$\overline{X^{-1}}x(p) \approx -\overline{X}x(p) \approx 0.$$

■

However, δI_pM is not an ideal of $\delta\Theta_pM$. As a matter of fact, $I \in \delta I_pM$ and if we define $X(q) := x^{-1}(x(q) + \delta u)$, with $u \in E \setminus \{0\}$, it follows that $X \in \delta\Theta_pM$ but $IX \notin \delta I_pM$.

We define a relation $\sim$ on $\delta\Theta_pM$ in the following way: given $X, Y \in \delta\Theta_pM$, we say that $X \sim Y$ if there exists $Z \in \delta I_pM$ with $X(p) = YZ(p)$.

Theorem 6.15 *$\sim$ is an equivalence relation.*

Proof. Clearly $X \sim X$ because $I \in \delta I_pM$.

Assume now that $X \sim Y$ and let $Z \in \delta I_p M$ be such that $X(p) = YZ(p)$. Therefore $p = Z^{-1}Y^{-1}X(p)$ and so it is also true that

$$Y(p) = X(X^{-1}YZ^{-1}Y^{-1}X)(p).$$

If we define $Z_1 := X^{-1}YZ^{-1}Y^{-1}X$, since

$$\overline{Z_1}x(p) \approx -\overline{X}x(p) + \overline{Y}x(p) - \overline{Z}x(p) - \overline{Y}x(p) + \overline{X}x(p) \approx 0,$$

it follows that $Z_1 \in \delta I_p M$ and since $Y(p) = XZ_1(p)$, $Y \sim X$.

Lastly, suppose that $X(p) = YZ_1(p)$ and $Y(p) = WZ_2(p)$, with $X, Y, W \in \delta\Theta_p M$ and $Z_1, Z_2 \in \delta I_p M$. Then $p = Y^{-1}WZ_2(p)$ and so

$$X(p) = YZ_1(p) = W(W^{-1}YZ_1Y^{-1}WZ_2)(p).$$

Define now $Z := W^{-1}YZ_1Y^{-1}WZ_2$. With similar calculations as done before, we conclude that $Z \in \delta I_p M$, which ends the proof. ■

Theorem 6.16 *There exists an isomorphism between $\delta\Theta_p M/\sim$ and E.*

Proof. Let

$$\begin{array}{rccl} \Phi : & \delta\Theta_p M/\sim & \to & E \\ & X & \mapsto & st(\overline{X}x(p)) \end{array}$$

The operator Φ is well defined because if $X \sim Y$ then

$$\Phi(X) = st(\overline{X}x(p)) = st(\overline{YZ}x(p))$$

$$= st(\overline{Y}x(p)) + st(\overline{Z}x(p)) = \Phi(Y),$$

for some $Z \in \delta I_p M$. Let us see that Φ is 1-1. Suppose that $\Phi(X) = \Phi(Y)$, for some $X, Y \in \delta\Theta_p M$. Then there exists an infinitesimal $\epsilon \in {}^*E$ with $xX(p) = xY(p) + \delta\epsilon$, which is equivalent to say that

$$X(p) = Y(Y^{-1}x^{-1}(xY(p) + \delta\epsilon)).$$

Let $Z(q) := Y^{-1}x^{-1}(xY(q)+\delta\epsilon)$. Then $Z \in \delta\Theta_p M$ with inverse $Z^{-1}(r) = Y^{-1}x^{-1}(xY(r)-\delta\epsilon)$ because

$$\begin{aligned}\overline{Z}(u) &= \frac{xY^{-1}x^{-1}(xYx^{-1}(u)+\delta\epsilon)-u}{\delta} \\ &= \frac{\underline{Y}^{-1}(\underline{Y}(u)+\delta\epsilon)-u}{\delta}\end{aligned}$$

and

$$\begin{aligned}D\overline{Z}_u &= \frac{D\underline{Y}^{-1}_{\underline{Y}(u)+\delta\epsilon}D\underline{Y}_u - I}{\delta} \\ &= \frac{(\delta D\overline{Y^{-1}}_{\underline{Y}(u)+\delta\epsilon}+I)(\delta D\overline{Y}_u+I)-I}{\delta} \\ &\approx D\overline{Y^{-1}}_{\underline{Y}(u)+\delta\epsilon} + D\overline{Y}_u\end{aligned}$$

which is a finite operator. Similarly, replacing $-\delta\epsilon$ for $\delta\epsilon$, we can prove that $\overline{Z^{-1}}$ is also SU-differentiable. Let us prove now that $Z \in \delta I_p M$.

$$\begin{aligned}\overline{Z}x(p) &= \frac{xY^{-1}x^{-1}(xY(p)+\delta\epsilon)-x(p)}{\delta} \\ &= \frac{xY^{-1}x^{-1}(xX(p))-x(p)}{\delta} \\ &= \overline{Y^{-1}X}x(p) \approx -\overline{Y}x(p) + \overline{X}x(p) \approx 0\end{aligned}$$

In conclusion, $X(p) = YZ(p)$ with $Z \in \delta I_p M$ and so $X \sim Y$. As done in Theorem 6.9, we can prove analogously that Φ is onto and linear. ∎

Theorem 6.17 *If $X, Y \in \delta\Theta_p M$ then $X \sim Y$ if and only if $X \equiv Y$.*

Proof. Suppose that $X \sim Y$ and let $Z \in \delta I_p M$ with $X(p) = YZ(p)$. Then

$$\begin{aligned}\overline{X}x(p) &= \frac{xX(p) - x(p)}{\delta} \\ &= \frac{xYZ(p) - x(p)}{\delta} \\ &= \overline{YZ}x(p) \approx \overline{Y}x(p) + \overline{Z}x(p) \\ &\approx \overline{Y}x(p).\end{aligned}$$

With similar calculations as done in the proof of the previous theorem we can prove the converse. ∎

6.3 Conjugation between δ-infinitesimal Transformations

Let M and N be two differentiable manifolds and $f : M \to N$ a standard diffeomorphism.

Given a δ-infinitesimal transformation on M, we can define a new one on N in the following way: for $X \in \delta\Theta_p M$ let $Y := fXf^{-1}$. Then $Y \in \delta\Theta_{f(p)}N$. In fact, Y is clearly an internal bijection with inverse $Y^{-1} = fX^{-1}f^{-1}$.

If $q \in ns({}^*N)$ then $Y(q) \approx q$ since

$$Y(q) \approx q \Leftrightarrow fXf^{-1}(q) \approx q \Leftrightarrow Xf^{-1}(q) \approx f^{-1}(q)$$

and $f^{-1}(q) \in ns({}^*M)$.

Finally let us prove that $\overline{Y}$ and $\overline{Y^{-1}}$ are both SU-differentiable. Let (U, x) be a chart at p and define $y := xf^{-1}|_V$, where V is an open set in N with $V \subseteq f(U)$ and $f(p) \in V$. Then (V, y) is a chart on N at $f(p)$ (simply note that y is compatible with the other charts on N). Moreover, we have seen that the SU-differentiability of $\overline{Y}$ does not depend of the choice of charts. So, if we fix this chart on N, we obtain

$$\overline{Y}(u) = \frac{yYy^{-1}(u) - u}{\delta} = \frac{xf^{-1}fXf^{-1}fx^{-1}(u) - u}{\delta} = \overline{X}(u)$$

and

$$\overline{Y^{-1}}(u) = \overline{X^{-1}}(u),$$

which are SU-differentiable.

Define then

$$\begin{array}{rccc} \mathfrak{F}_p f: & T_pM & \rightarrow & T_{f(p)}N \\ & (p, st(\overline{X}x(p))) & \mapsto & (f(p), st(\overline{fXf^{-1}}yf(p))) \end{array}$$

Let us begin by proving that this function is well defined. For $X, Y \in \delta\Theta_p M$ with $\overline{X}x(p) \approx \overline{Y}x(p)$ we have $\overline{fXf^{-1}}yf(p) \approx \overline{fYf^{-1}}yf(p)$. In fact

$$\overline{X}x(p) \approx \overline{Y}x(p) \Leftrightarrow \frac{xX(p) - xY(p)}{\delta} \approx 0.$$

On the other hand

$$\overline{fXf^{-1}}yf(p) - \overline{fYf^{-1}}yf(p) = \frac{yfX(p) - yfY(p)}{\delta} = \frac{xX(p) - xY(p)}{\delta} \approx 0.$$

If we choose $y := xf^{-1}|_V$ then

$$\mathfrak{F}_p f(p, st(\overline{X}x(p))) = (f(p), st(\overline{X}x(p))).$$

With simple calculations we can prove the following theorems:

Theorem 6.18 *Let $f : M \rightarrow N$ and $g : N \rightarrow R$ be two diffeomorphisms. Then is well defined $\mathfrak{F}_p gf : T_pM \rightarrow T_{gf(p)}R$ and $\mathfrak{F}_p gf = \mathfrak{F}_{f(p)} g \mathfrak{F}_p f$.*

Theorem 6.19 *The function $\mathfrak{F}_p f$ is linear.*

Theorem 6.20 *The function $\mathfrak{F}_p f$ is invertible with inverse $(\mathfrak{F}_p f)^{-1} = \mathfrak{F}_{f(p)} f^{-1}$.*

We can generalize the previous definition to the tangent bundle of a manifold. Let $f : M \rightarrow N$ be a diffeomorphism and define

$$\begin{array}{rccc} \mathfrak{F} f: & TM & \rightarrow & TN \\ & (p, st(\overline{X}x(p))) & \mapsto & (f(p), st(\overline{fXf^{-1}}yf(p))) \end{array}$$

Similarly we have

Theorem 6.21 *The following is verified*

1. *The function $\mathfrak{F} f$ is invertible and $(\mathfrak{F} f)^{-1} = \mathfrak{F} f^{-1}$;*
2. *If $f = I$ then $\mathfrak{F} f = I$;*
3. *$\mathfrak{F} gf = \mathfrak{F} g \mathfrak{F} f$;*
4. *The following diagram is commutative*

$$\begin{array}{ccccc} & & \mathfrak{F} f & & \\ & TM & \longrightarrow & TN & \\ \pi_M & \downarrow & & \downarrow & \pi_N \\ & M & \longrightarrow & N & \\ & & f & & \end{array}$$

i.e., $f\pi_M = \pi_N \mathfrak{F} f$, where π_M and π_N are the canonical projections.

6.4 The Differential of a Function

Let M and N be two differentiable manifolds. With a function $f : M \to N$ of class C^k and for a fixed $p \in M$ we can associate a linear operator $T_p f : T_p M \to T_{f(p)} N$ that maps tangent vectors into tangent vectors. Indeed, define

Definition 6.22 *The **differential** of f at p is the function*

$$\begin{array}{rccc} T_p f : & T_p M & \to & T_{f(p)} N \\ & (p, st(\overline{X} x(p))) & \mapsto & (f(p), D(yfx^{-1})_{x(p)} st(\overline{X} x(p))) \end{array}$$

where (U, x) is a chart on M at p and (V, y) a chart on N at $f(p)$, with $f(U) \subseteq V$.

The δ-infinitesimal transformation associated on $T_{f(p)} N$ is

$$Y(q) := y^{-1}(y(q) + \delta D(yfx^{-1})_{x(p)} st(\overline{X} x(p))).$$

Since f is a function of class C^k, $T_p f$ is a function of class C^{k-1}. If f is the identity function then $T_p f$ is also the identity function.

Theorem 6.23 *Let $f : M \to N$ and $g : N \to P$ be two functions of class C^k. Then $T_p(gf) = T_{f(p)}gT_pf$.*

Proof. Let (U, x) be a chart at p, (W, z) a chart at $f(p)$ and (V, y) another chart at $gf(p)$, with $f(U) \subseteq W$ and $g(W) \subseteq V$. Then

$$\begin{aligned} T_pgf(p, st(\overline{X}x(p))) &= (gf(p), D(ygfx^{-1})_{x(p)}st(\overline{X}x(p))) \\ &= (gf(p), D(ygz^{-1})_{zf(p)}D(zfx^{-1})_{x(p)}st(\overline{X}x(p))) \\ &= T_{f(p)}g(f(p), D(zfx^{-1})_{x(p)}st(\overline{X}x(p))) \\ &= T_{f(p)}gT_pf(p, st(\overline{X}x(p))) \end{aligned}$$

■

The following properties hold:

Theorem 6.24 *The operador T_pf is linear.*

Theorem 6.25 *If f is a diffeomorphism then T_pf is an isomorphism and $(T_pf)^{-1} = T_{f(p)}f^{-1}$.*

Proof. The inverse of T_pf is

$$(T_pf)^{-1}(f(p), st(\overline{Y}yf(p))) = (p, D(xf^{-1}y^{-1})_{yf(p)}st(\overline{Y}yf(p))).$$

■

Theorem 6.26 *The following diagram is commutative.*

$$\begin{array}{ccccc} & & T_pf & & \\ & T_pM & \longrightarrow & T_{f(p)}N & \\ \pi_M & \downarrow & & \downarrow & \pi_N \\ & M & \longrightarrow & N & \\ & & f & & \end{array}$$

where π_M and π_N are the canonical projections.

6.5 Directional Derivative of a Function

Let M be a differentiable manifold, F a normed space and $f : M \to F$ a function of class C^1.

Definition 6.27 *For $p \in M$, we define the **directional derivative** of f at p as being*

$$\begin{array}{cccc} Df_p : & T_pM & \to & F \\ & (p, st(\overline{X}x(p))) & \mapsto & st\left(\dfrac{fX(p) - f(p)}{\delta}\right) \end{array}$$

Observe that, for some $\eta \approx 0$,

$$\begin{aligned} Df_p(p, st(\overline{X}x(p))) &= st\left(\frac{(fx^{-1})xX(p) - (fx^{-1})x(p)}{\delta}\right) \\ &= st[D(fx^{-1})_{x(p)}\overline{X}x(p) + |\overline{X}x(p)|\eta] \\ &= D(fx^{-1})_{x(p)}st(\overline{X}x(p)) \end{aligned}$$

Consequently, Df_p is well defined, *i.e.*, if

$$(p, st(\overline{X}x(p))) \equiv (p, st(\overline{Y}x(p)))$$

then

$$Df_p(p, st(\overline{X}x(p)))) = Df_p(p, st(\overline{Y}x(p))).$$

As one might expect,

Theorem 6.28 *The operator Df_p is linear.*

6.6 Functionals defined on a Manifold

In this section we will study some properties of functionals of class C^∞ on M.

Definition 6.29 *Let $p \in M$, (U, x) a chart for M whose domain contains p and $X \in \delta\Theta_p M$. We define*

$$\begin{array}{rccl} X': & C^\infty(M) & \to & \mathbb{R} \\ & f & \mapsto & st\left(\dfrac{fX(p) - f(p)}{\delta}\right) \end{array}$$

*The function X' will be called the **derivative** of X at p.*

The function X' is well defined since

$$\frac{fX(p) - f(p)}{\delta} \approx D(fx^{-1})_{x(p)}\overline{X}x(p) \in fin({}^*\mathbb{R})$$

and so

$$X'(f) = D(fx^{-1})_{x(p)} st(\overline{X}x(p)).$$

The following properties hold ([SL76]):

For all $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. $X'(f + g) = X'(f) + X'(g)$;
2. $X'(af) = aX'(f)$;
3. $X'(fg) = f(p)X'(g) + g(p)X'(f)$.

   To these properties we add a fourth:

4. $X'(f/g) = \dfrac{g(p)X'(f) - f(p)X'(g)}{g^2(p)}$ if $g(p) \neq 0$. In fact,

$$X'\left(\frac{1}{g}\right) = st\left(\frac{\frac{1}{gX(p)} - \frac{1}{g(p)}}{\delta}\right) = -st\left(\frac{gX(p) - g(p)}{\delta g(p) gX(p)}\right).$$

   Since g is a continuous function and $X(p) \approx p$, it follows that $st(gX(p)) = g(p) \neq 0$. Hence

$$X'\left(\frac{1}{g}\right) = -X'(g)\frac{1}{g^2(p)}$$

   and

$$X'\left(\frac{f}{g}\right) = X'\left(f \cdot \frac{1}{g}\right) = \frac{g(p)X'(f) - f(p)X'(g)}{g^2(p)}.$$

The first two conditions prove that X' is a linear operator of $C^\infty(M)$ to $\mathbb{R}$. The third condition justifies the term *derivative* (the "Leibniz rule").

The set of derivatives at $p \in M$ will be denoted by D_pM:

$$D_pM := \{X' : C^\infty(M) \to \mathbb{R} \,|\, X'(f) = st\left(\frac{fX(p) - f(p)}{\delta}\right) \wedge X \in \delta\Theta_pM\}$$

If we define for $X', Y' \in D_pM$, $f \in C^\infty(M)$ and $a \in \mathbb{R}$,

$$(X' + Y')(f) = X'(f) + Y'(f)$$

$$(aX')(f) = aX'(f)$$

the set D_pM becomes a real linear space. If $I : M \to M$ denotes the identity function then $I'(f) = 0$, for every $f \in C^\infty(M)$. Moreover, $-(X') = (-X)'$ (recall the scalar multiplication on $\delta\Theta_pM$). In fact,

$$X'(f) + (-X)'(f) = D(fx^{-1})_{x(p)}st(\overline{X}x(p)) + D(fx^{-1})_{x(p)}st(\overline{-X}x(p)) = 0.$$

Observe that we can also write

$$(X' + Y')(f) = D(fx^{-1})_{x(p)}st\,\overline{XY}x(p)$$

and

$$(aX')(f) = D(fx^{-1})_{x(p)}st\,\overline{aX}x(p).$$

From the previous observations it follows that

Theorem 6.30 *[SL76] It is true that for $X, Y \in \delta\Theta_pM$ and $f \in C^\infty(M)$,*

$$(XY)'(f) = X'(f) + Y'(f).$$

Theorem 6.31 *For $X \in \delta\Theta_pM, f \in C^\infty(M)$ and $a \in \mathbb{R}$,*

$$(aX)'(f) = aX'(f).$$

Theorem 6.32 *The following properties hold:*

1. *If f is constant then $X'(f) = 0$;*
2. *If $f(p) = g(p) = 0$ then $X'(fg) = 0$;*
3. *If $f = g$ in a neighbourhood of p then $X'(f) = X'(g)$.*

Proof. The second condition follows from the Leibniz rule. The other two are clear. ∎

Theorem 6.33 *There exists an isomorphism between D_pM and E.*

Proof. Let

$$\begin{array}{rlcl} \Omega: & D_pM & \to & E \\ & X' & \mapsto & st(\overline{X}x(p)) \end{array}$$

It is clear that the operator Ω is linear. Let us see that it is bijective.

1. it is 1-1: Let $X', Y' \in D_pM$ with $\Omega(X') = \Omega(Y')$, *i.e.*,

   $$st(\overline{X}x(p)) = st(\overline{Y}x(p)).$$

   Now let $f \in C^\infty(M)$. Then

   $$\begin{aligned} X'(f) &= D(fx^{-1})_{x(p)} st(\overline{X}x(p)) \\ &= D(fx^{-1})_{x(p)} st(\overline{Y}x(p)) \\ &= Y'(f). \end{aligned}$$

   Thus $X' = Y'$.

2. it is onto: Fix $u \in E$ and define $X(q) := x^{-1}(x(q) + \delta u)$. Then $X \in \delta\Theta_pM$ and $\Omega(X') = u$.

∎

Theorem 6.34 *The sets T_pM and D_pM are isomorphic.*

Proof. Follows from the previous theorem and from Theorem 6.9. ∎

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